CHAPTER 4

Convergence of Fourier Series

4.1. Pointwise Convergence of Fourier Series

As before, E denotes the collection of all functions $f : [-\pi, \pi] \to \mathbb{C}$ which are piecewise continuous on the interval $[-\pi, \pi]$. Each of these functions has at most a finite number of points of discontinuity, at each of which the function need not be defined but must have one sided limits which are finite. Again we adopt the convention that any two functions $f, g \in E$ are considered equal if $f(x) = g(x)$ for every $x \in [-\pi, \pi]$ with at most a finite number of exceptions.

If we assume that the infinite orthonormal system

$$\left\{ \frac{1}{\sqrt{2}}, \sin x, \cos x, \sin 2x, \cos 2x, \sin 3x, \cos 3x, \ldots \right\}$$

is closed in E, then for every $f \in E$ with Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

we have convergence in norm. In other words,

$$\lim_{m\to\infty} \left\| f(x) - \left(\frac{a_0}{2} + \sum_{n=1}^{m} (a_n \cos nx + b_n \sin nx) \right) \right\| = 0,$$

or

$$\lim_{m\to\infty} \int_{-\pi}^{\pi} \left| f(x) - \left(\frac{a_0}{2} + \sum_{n=1}^{m} (a_n \cos nx + b_n \sin nx) \right) \right|^2 \mathrm{d}x = 0.$$

In this chapter, we study conditions under which

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

To do this, we need to restrict our discussion to a more special set of functions in E. Accordingly, we define E' to denote the collection of functions $f \in E$ that satisfy the following two conditions:

(i) For every $x \in [-\pi, \pi)$, the limit

$$\lim_{h\to 0^+} \frac{f(x+h) - f(x+0)}{h}$$

exists and is finite. Here $f(x+0)$ denotes the right hand limit of f at x.

(ii) For every $x \in (-\pi, \pi]$, the limit

$$\lim_{h\to 0^+} \frac{f(x-0) - f(x-h)}{h}$$

exists and is finite. Here $f(x-0)$ denotes the left hand limit of f at x.

In other words, appropriate one sided derivatives exist at every point in $[-\pi, \pi]$.

THEOREM 4.1 (Dirichlet's theorem). *Suppose that $f \in E'$. Then for every $x \in (-\pi, \pi)$, the Fourier series of f converges to the value*

$$\frac{f(x-0)+f(x+0)}{2}.$$

Furthermore, at each endpoint $x = \pm\pi$, the Fourier series of f converges to the value

$$\frac{f(\pi-0)+f(-\pi+0)}{2}.$$

REMARK. Recall that we may change the value of a function $f \in E$ at finitely many points without affecting the Fourier series. If we then interpret such a function f to be a 2π-periodic function defined on the real line $\mathbb{R}$, then the Fourier series of f converges to the value f at every point of continuity of f, and converges to the average value of the left hand and right hand limits of f at every point of discontinuity of f.

Assume, without loss of generality, that $f \in E'$ is defined on $\mathbb{R}$ and is 2π-periodic. Let $x \in \mathbb{R}$ be chosen and fixed. For every such $f \in E'$, consider the partial sum

$$S_m(x) = \frac{a_0}{2} + \sum_{n=1}^{m}(a_n \cos nx + b_n \sin nx)$$

of the Fourier series of f. Using the definitions for the Fourier coefficients, we have

$$\begin{aligned}
S_m(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,\mathrm{d}y + \sum_{n=1}^{m}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\cos ny\cos nx\,\mathrm{d}y + \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\sin ny\sin nx\,\mathrm{d}y\right)\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\left(\frac{1}{2} + \sum_{n=1}^{m}(\cos ny\cos nx + \sin ny\sin nx)\right)\mathrm{d}y\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\left(\frac{1}{2} + \sum_{n=1}^{m}\cos n(y-x)\right)\mathrm{d}y.
\end{aligned}$$

The substitution $t = y - x$ then gives

$$S_m(x) = \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} f(x+t)\left(\frac{1}{2} + \sum_{n=1}^{m}\cos nt\right)\mathrm{d}t.$$

Observe now that the integrand is 2π-periodic in the variable t. It follows from periodicity that

$$S_m(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\left(\frac{1}{2} + \sum_{n=1}^{m}\cos nt\right)\mathrm{d}t.$$

We next study the term

$$\frac{1}{2} + \sum_{n=1}^{m}\cos nt.$$

Using the identity $\cos nt \sin\frac{1}{2}t = \frac{1}{2}(\sin(n+\frac{1}{2})t - \sin(n-\frac{1}{2})t)$ for every $n = 1, \ldots, m$, we conclude that

$$\frac{1}{2} + \sum_{n=1}^{m}\cos nt = \frac{\sin(m+\frac{1}{2})t}{2\sin\frac{1}{2}t}. \tag{4.1}$$

It follows that

$$S_m(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\frac{\sin(m+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,\mathrm{d}t.$$

To prove Dirichlet's theorem, we need to show that

$$\lim_{m\to\infty} S_m(x) = \frac{f(x-0)+f(x+0)}{2}.$$

We therefore split the interval of integration into two accordingly, so that

$$S_m(x) = \frac{1}{\pi}\int_{-\pi}^{0} f(x+t)\frac{\sin(m+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,\mathrm{d}t + \frac{1}{\pi}\int_{0}^{\pi} f(x+t)\frac{\sin(m+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,\mathrm{d}t.$$

Clearly it suffices to show that

$$\lim_{m\to\infty} \frac{1}{\pi}\int_0^\pi f(x+t)\frac{\sin(m+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,\mathrm{d}t = \frac{f(x+0)}{2}, \tag{4.2}$$

and that

$$\lim_{m\to\infty} \frac{1}{\pi}\int_{-\pi}^0 f(x+t)\frac{\sin(m+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,\mathrm{d}t = \frac{f(x-0)}{2}. \tag{4.3}$$

The deduction of (4.2) and (4.3) are rather similar, so we shall only discuss (4.2). The idea is to replace the term $f(x+t)$ by $f(x+0)$. We have

$$\int_0^\pi f(x+t)\frac{\sin(m+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,\mathrm{d}t - \int_0^\pi f(x+0)\frac{\sin(m+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,\mathrm{d}t = \int_0^\pi \frac{f(x+t)-f(x+0)}{2\sin\frac{1}{2}t}\sin(m+\tfrac{1}{2})t\,\mathrm{d}t.$$

The second term on the left hand side can be handled quite easily. We have

$$\int_0^\pi f(x+0)\frac{\sin(m+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,\mathrm{d}t = f(x+0)\int_0^\pi \frac{\sin(m+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,\mathrm{d}t = \frac{\pi f(x+0)}{2},$$

the last equality being a simple consequence of the trigonometric identity (4.1) and the observation that

$$\int_0^\pi \cos nt\,\mathrm{d}t = 0, \quad \text{if } n = 1,\ldots,m.$$

Hence (4.2) will follow if we can show that

$$\lim_{m\to\infty}\int_0^\pi \frac{f(x+t)-f(x+0)}{2\sin\frac{1}{2}t}\sin(m+\tfrac{1}{2})t\,\mathrm{d}t = 0.$$

Note that the function

$$g(t) = \frac{f(x+t)-f(x+0)}{2\sin\frac{1}{2}t}$$

is piecewise continuous on $(0,\pi]$, and the limit

$$\lim_{t\to 0^+} g(t) = \lim_{t\to 0^+}\frac{f(x+t)-f(x+0)}{2\sin\frac{1}{2}t} = \lim_{t\to 0^+}\left(\frac{f(x+t)-f(x+0)}{t}\right)\left(\frac{t}{2\sin\frac{1}{2}t}\right)$$

exists. Hence the function $g(t)$ is piecewise continuous on $[0,\pi]$. The deduction of (4.2) will be complete if we can establish the following technical result which shows that high oscillation eschews a function.

THEOREM 4.2. *Suppose that g is a piecewise continuous function on $[0,\pi]$. Then*

$$\lim_{m\to\infty}\int_0^\pi g(t)\sin(m+\tfrac{1}{2})t\,\mathrm{d}t = 0.$$

PROOF. For every $m \in \mathbb{N}$, we have

$$\int_0^\pi g(t)\sin(m+\tfrac{1}{2})t\,\mathrm{d}t = \int_0^\pi g(t)\cos\tfrac{1}{2}t\sin mt\,\mathrm{d}t + \int_0^\pi g(t)\sin\tfrac{1}{2}t\cos mt\,\mathrm{d}t. \tag{4.4}$$

Let $h_1 \in E$ be defined by

$$h_1(t) = \begin{cases} g(t)\cos\frac{1}{2}t, & \text{if } 0 \leqslant t \leqslant \pi, \\ 0, & \text{if } -\pi \leqslant t < 0. \end{cases}$$

Then

$$\frac{1}{\pi}\int_0^\pi g(t)\cos\tfrac{1}{2}t\sin mt\,\mathrm{d}t = \frac{1}{\pi}\int_{-\pi}^\pi h_1(t)\sin mt\,\mathrm{d}t = \langle h_1, \sin mt\rangle \to 0, \quad \text{as } m\to\infty, \tag{4.5}$$

by the Riemann–Lebesgue lemma. Similarly, let $h_2 \in E$ be defined by

$$h_2(t) = \begin{cases} g(t)\sin\frac{1}{2}t, & \text{if } 0 \leqslant t \leqslant \pi, \\ 0, & \text{if } -\pi \leqslant t < 0. \end{cases}$$

Then

$$\frac{1}{\pi}\int_0^\pi g(t)\sin\tfrac{1}{2}t\cos mt\,\mathrm{d}t = \frac{1}{\pi}\int_{-\pi}^{\pi} h_2(t)\cos mt\,\mathrm{d}t = \langle h_2, \cos mt\rangle \to 0, \quad \text{as } m\to\infty, \tag{4.6}$$

by the Riemann–Lebesgue lemma. The result follows on combining (4.4)–(4.6). ◯

One of the applications of Dirichlet's theorem is on the evaluation of various infinite series. We give here a few simple examples.

Example 4.1.1. In Example 3.1.1, we have

$$x \sim \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx.$$

For $x=\pi/2$, Dirichlet's theorem gives

$$\frac{\pi}{2} = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin\frac{n\pi}{2} = 2\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots\right), \quad \text{so that} \quad 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots = \frac{\pi}{4}.$$

Example 4.1.2. In Example 3.1.2, we have

$$|x| \sim \frac{\pi}{2} - \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{4}{\pi n^2}\cos nx.$$

For $x=0$, Dirichlet's theorem gives

$$\frac{\pi}{2} = \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{4}{\pi n^2} = \frac{4}{\pi}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots\right), \quad \text{so that} \quad 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots = \frac{\pi^2}{8}.$$

Example 4.1.3. In Example 3.1.3, we have

$$\operatorname{sgn}(x) = \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{4}{\pi n}\sin nx.$$

For $x=\pi/2$, Dirichlet's theorem gives

$$1 = \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{4}{\pi n}\sin\frac{n\pi}{2} = \frac{4}{\pi}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots\right), \quad \text{so that} \quad 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots = \frac{\pi}{4}.$$

Note that this is the same result as in Example 4.1.1.

Example 4.1.4. In Example 3.1.4, we have

$$x^2 \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx.$$

For $x=0$, Dirichlet's theorem gives

$$0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2} = \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}, \quad \text{so that} \quad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

For $x=\pi$, Dirichlet's theorem gives

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos n\pi = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}, \quad \text{so that} \quad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

4.2. Introduction to Uniform Convergence

In this section, we give a brief introduction to the notion of uniform convergence and discuss a number of results that we need.

Suppose that $f_m : X \to \mathbb{C}$ is a sequence of functions defined on a non-empty set $X \subseteq \mathbb{R}$. The sequence f_m is said to converge pointwise on X if there exists a function $f : X \to \mathbb{C}$ such that, given any $\epsilon > 0$, for every $x \in X$, there exists $N = N(\epsilon, x)$ such that

$$|f_m(x) - f(x)| < \epsilon, \quad \text{if } m \geqslant N(\epsilon, x).$$

Note here that the choice of N may depend on the choice of $x \in X$ as well as the value of $\epsilon > 0$. In essence, we are studying the convergence of the sequence $f_m(x)$ for each $x \in X$ separately.

We have the General principle of convergence, that a sequence f_m converges pointwise on X if and only if, given any $\epsilon > 0$, for every $x \in X$, there exists $N = N(\epsilon, x)$ such that

$$|f_m(x) - f_k(x)| < \epsilon, \quad \text{if } m > k \geqslant N(\epsilon, x).$$

For more details and the proof, the interested reader is referred to Chapter 2 in *Fundamentals of Analysis.*

EXAMPLE 4.2.1. Let $X = [0, 1]$. For every $m \in \mathbb{N}$ and every $x \in [0, 1]$, let $f_m(x) = x^m$. Then the sequence f_m converges pointwise on X to the function $f : X \to \mathbb{C}$, where $f(x) = 0$ if $0 \leqslant x < 1$ and $f(1) = 1$. Note that each of the functions f_m is continuous on X, but the limit function f is not continuous on X. Hence the continuity property of the functions f_m is not carried over to the limit function f.

To carry over certain properties of the individual functions of a sequence to the limit function, we need a type of convergence which is stronger than pointwise convergence. Suppose that $f_m : X \to \mathbb{C}$ is a sequence of functions defined on a non-empty set $X \subseteq \mathbb{R}$. The sequence f_m is said to converge uniformly on X if there exists a function $f : X \to \mathbb{C}$ such that, given any $\epsilon > 0$, there exists $N = N(\epsilon)$, independent of $x \in X$, such that

$$|f_m(x) - f(x)| < \epsilon, \quad \text{if } m \geqslant N(\epsilon) \text{ and } x \in X.$$

We shall first of all extend the General principle of convergence to the case of uniform convergence.

THEOREM 4.3 (General principle of uniform convergence). *Let $f_m : X \to \mathbb{C}$ be a sequence of functions defined on a non-empty set $X \subseteq \mathbb{R}$. Then f_m converges uniformly on X if and only if, given any $\epsilon > 0$, there exists $N = N(\epsilon)$, independent of $x \in X$, such that*

$$|f_m(x) - f_k(x)| < \epsilon, \quad \textit{if } m > k \geqslant N(\epsilon) \textit{ and } x \in X.$$

PROOF. ($\Rightarrow$) Suppose that f_m converges uniformly on X. Then there exists a function $f : X \to \mathbb{C}$ such that, given any $\epsilon > 0$, there exists $N = N(\epsilon)$, independent of $x \in X$, such that

$$|f_m(x) - f(x)| < \tfrac{1}{2}\epsilon, \quad \text{if } m \geqslant N(\epsilon) \text{ amd } x \in X.$$

It follows that

$$|f_m(x) - f_k(x)| \leqslant |f_m(x) - f(x)| + |f_k(x) - f(x)| < \epsilon, \quad \text{if } m > k \geqslant N(\epsilon) \text{ and } x \in X.$$

($\Leftarrow$) It follows from the General principle of convergence that f_m converges pointwise on X, to a function f, say. We shall show that f_m converges to f uniformly on X. Given any $\epsilon > 0$, there exists $N = N(\epsilon)$, independent of $x \in X$, such that

$$|f_k(x) - f_m(x)| < \tfrac{1}{2}\epsilon, \quad \text{if } k > m \geqslant N(\epsilon) \text{ and } x \in X.$$

Hence

$$|f(x) - f_m(x)| = \lim_{k\to\infty} |f_k(x) - f_m(x)| \leqslant \tfrac{1}{2}\epsilon, \quad \text{if } m \geqslant N(\epsilon) \text{ and } x \in X,$$

so that

$$|f_m(x) - f(x)| < \epsilon, \quad \text{if } m \geqslant N(\epsilon) \text{ and } x \in X.$$

Hence f_m converges uniformly on X. ◯

We use the General principle of uniform convergence where the sequence f_m is the sequence of partial sums of a Fourier series. Suppose that $u_n : X \to \mathbb{C}$ is a sequence of functions defined on a non-empty set $X \subseteq \mathbb{R}$. The series

$$\sum_{n=1}^{\infty} u_n(x)$$

is said to converge uniformly on X if the sequence of partial sums

$$S_m(x) = \sum_{n=1}^{m} u_n(x)$$

converges uniformly on X. The General principle of uniform convergence, applied to this case, then says that given any $\epsilon > 0$, there exists $N = N(\epsilon)$, independent of $x \in X$, such that

$$\left| \sum_{n=k+1}^{m} u_n(x) \right| < \epsilon, \quad \text{if } m > k \geqslant N(\epsilon) \text{ and } x \in X.$$

We have the following analogue of the Comparison test.

THEOREM 4.4 (Weierstrass M-test). *Suppose that $u_n : X \to \mathbb{C}$ is a sequence of functions defined on a non-empty set $X \subseteq \mathbb{R}$. Suppose further that for every $n \in \mathbb{N}$, there exists a real constant M_n such that the series*

$$\sum_{n=1}^{\infty} M_n$$

is convergent, and that $|u_n(x)| \leqslant M_n$ for every $x \in X$. Then the series

$$\sum_{n=1}^{\infty} u_n(x)$$

converges uniformly on X.

PROOF. Given any $\epsilon > 0$, it follows from the General principle of convergence (for series) that there exists $N = N(\epsilon)$ such that

$$\sum_{n=k+1}^{m} M_n < \epsilon, \quad \text{if } m > k \geqslant N(\epsilon).$$

It follows that

$$\left| \sum_{n=k+1}^{m} u_n(x) \right| \leqslant \sum_{n=k+1}^{m} M_n < \epsilon, \quad \text{if } m > k \geqslant N(\epsilon) \text{ and } x \in X.$$

It now follows from the General principle of uniform convergence that the series

$$\sum_{n=1}^{\infty} u_n(x)$$

converges uniformly on X. ◯

To complement the observation made in Example 4.2.1, we establish the following result.

THEOREM 4.5. *Suppose that a sequence of functions $f_m : X \to \mathbb{C}$ defined on a non-empty set $X \subseteq \mathbb{R}$ converges uniformly on X to a function $f : X \to \mathbb{C}$. Suppose further that $c \in X$ and that the function f_m is continuous at c for every $m \in \mathbb{N}$. Then the function f is continuous at c.*

REMARK. In other words, uniform convergence of functions preserves continuity.

PROOF OF THEOREM 4.5. Suppose that $\epsilon > 0$ is given. Since f_m converges uniformly on X to f, there exists $m \in \mathbb{N}$, independent of $x \in X$, such that

$$|f_m(x) - f(x)| < \tfrac{1}{3}\epsilon, \quad \text{if } x \in X.$$

Since f_m is continuous at c, there exists $\delta > 0$ such that

$$|f_m(x) - f_m(c)| < \tfrac{1}{3}\epsilon, \quad \text{if } |x - c| < \delta.$$

It follows that

$$|f(x) - f(c)| \leqslant |f(x) - f_m(x)| + |f_m(x) - f_m(c)| + |f_m(c) - f(c)| < \epsilon, \quad \text{if } |x - c| < \delta.$$

Hence f is continuous at c. ◯

4.3. Uniform Convergence of Fourier Series

Let us now return to our discussion of Fourier series. Suppose that $f \in E$, with Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

Clearly for every $m \in \mathbb{N}$, the partial sum

$$S_m(x) = \frac{a_0}{2} + \sum_{n=1}^{m}(a_n \cos nx + b_n \sin nx)$$

is a continuous function on the closed interval $[-\pi, \pi]$, with $S_m(-\pi) = S_m(\pi)$. It then follows from Theorem 4.5 that if the sequence S_m converges to the function f, then it is necessary for f to be continuous on $[-\pi, \pi]$, with $f(-\pi) = f(\pi)$. In fact, we have the following nice result.

THEOREM 4.6. *Suppose that $f \in E$ is continuous on $[-\pi, \pi]$ and satisfies $f(-\pi) = f(\pi)$. Suppose further that $f' \in E$, so that $f \in E'$ in particular. Then the Fourier series of f converges uniformly to f on $[-\pi, \pi]$.*

PROOF. Suppose that f has Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

and that the derivative f' has Fourier series

$$f'(x) \sim \frac{\alpha_0}{2} + \sum_{n=1}^{\infty}(\alpha_n \cos nx + \beta_n \sin nx).$$

From the definition of Fourier coefficients, we have

$$\alpha_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\,\mathrm{d}x = \frac{f(\pi) - f(-\pi)}{\pi} = 0,$$

since $f(-\pi) = f(\pi)$. Using $f(-\pi) = f(\pi)$ again, we see that for every $n \in \mathbb{N}$, we have

$$\alpha_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\Big[f(x)\cos nx\Big]_{-\pi}^{\pi} + \frac{n}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x = nb_n,$$

and

$$\beta_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\Big[f(x)\sin nx\Big]_{-\pi}^{\pi} - \frac{n}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = -na_n.$$

It follows that f' has Fourier series

$$f'(x) \sim \sum_{n=1}^{\infty}(nb_n \cos nx - na_n \sin nx).$$

Consider now the series

$$\sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2)^{\frac{1}{2}} = \sum_{n=1}^{\infty}\frac{1}{n}(|\alpha_n|^2 + |\beta_n|^2)^{\frac{1}{2}}.$$

Using the Cauchy–Schwarz inequality

$$\left|\sum_{n=1}^{\infty} x_n\overline{y_n}\right| \leqslant \left(\sum_{n=1}^{\infty}|x_n|^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}|y_n|^2\right)^{\frac{1}{2}}, \tag{4.7}$$

we have

$$\sum_{n=1}^{\infty}(|a_n|^2+|b_n|^2)^{\frac{1}{2}} = \sum_{n=1}^{\infty}\frac{1}{n}(|\alpha_n|^2+|\beta_n|^2)^{\frac{1}{2}} \leqslant \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}(|\alpha_n|^2+|\beta_n|^2)\right)^{\frac{1}{2}}.$$

From Example 4.1.4, we have

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

On the other hand, it follows from Bessel's inequality applied to the function f' that

$$\sum_{n=1}^{\infty}(|\alpha_n|^2+|\beta_n|^2) \leqslant \|f'\|^2.$$

It follows that the series

$$\sum_{n=1}^{\infty}(|a_n|^2+|b_n|^2)^{\frac{1}{2}} \tag{4.8}$$

is convergent. Next, note that for every $n \in \mathbb{N}$ and $x \in [-\pi,\pi]$, we have

$$|a_n \cos nx| \leqslant (|a_n|^2+|b_n|^2)^{\frac{1}{2}} \quad \text{and} \quad |b_n \sin nx| \leqslant (|a_n|^2+|b_n|^2)^{\frac{1}{2}}.$$

The uniform convergence of the Fourier series of f now follows from the Weierstrass M-test together with the convergence of the series (4.8). That the Fourier series of f converges pointwise to f already is a consequence of Dirichlet's theorem. ◯

Remark. Consider the set $\mathbb{C}^m = \{\mathbf{x}=(x_1,\ldots,x_m) : x_1,\ldots,x_m \in \mathbb{C}\}$ of all m-tuples of complex numbers. It is easy to check that $\mathbb{C}^m$ forms a vector space over $\mathbb{C}$ under coordinatewise addition and scalar multiplication. It is also not difficult to show that

$$\langle \mathbf{x},\mathbf{y}\rangle = \sum_{n=1}^{m} x_n\overline{y_n}$$

gives an inner product on $\mathbb{C}^m$. The Cauchy–Schwarz inequality in this case is then given by

$$\left|\sum_{n=1}^{m} x_n\overline{y_n}\right| \leqslant \left(\sum_{n=1}^{m}|x_n|^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{m}|y_n|^2\right)^{\frac{1}{2}}. \tag{4.9}$$

Note that this is a finite version of the inequality (4.7). Suppose now that the two series on the right hand side of (4.7) both converge. Then using the Comparison test, one can show that the series on the left hand side of (4.7) also converges. On the other hand, it follows from (4.9) that

$$\left|\sum_{n=1}^{m} x_n\overline{y_n}\right| \leqslant \left(\sum_{n=1}^{\infty}|x_n|^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}|y_n|^2\right)^{\frac{1}{2}}, \quad \text{if } m \in \mathbb{N}.$$

The inequality (4.7) now follows on taking the limit on the left hand side as $m \to \infty$.

Suppose that a function $f \in E$ has discontinuities in the interval $[-\pi,\pi]$. If a closed subinterval $[a,b]$ contains a point of discontinuity of f, then clearly the Fourier series of f cannot converge uniformly on $[a,b]$. We next show the following complementary result.

Theorem 4.7. *Suppose that $f \in E$ is continuous on the interval $[-\pi,\pi]$, apart from finitely many points of discontinuity $d_1,\ldots,d_k \in (-\pi,\pi]$, with the convention that f is continuous at π if and only if $f(-\pi)=f(\pi)$. Suppose further that $f' \in E$, and that a subinterval $[a,b]$ of $[-\pi,\pi]$ does not contain any of these points of discontinuity of f. Then the Fourier series of f converges uniformly to f on $[a,b]$.*

Our study of this result is based on the case of a rather simple function with only one point of discontinuity in the interval $[-\pi,\pi]$. For the general case, we shall attempt to write a function with finitely many points of discontinuity in terms of a function which has no point of discontinuity in the interval,$[-\pi,\pi]$ and a sum of finitely many functions, each of which described in terms of our simple function and having only one point of discontinuity in the interval $(-\pi,\pi]$.

THEOREM 4.8. *For every $c \in (0,\pi)$, the Fourier series of the function $\phi : [-\pi,\pi] \to \mathbb{C}$, given by*

$$\phi(x) = \begin{cases} x, & \text{if } -\pi < x < \pi, \\ 0, & \text{if } x = \pm\pi, \end{cases}$$

converges uniformly to ϕ on the closed interval $[-c,c]$.

PROOF. It follows from Example 3.1.1 and Dirichlet's theorem that

$$\phi(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx,$$

so that the Fourier series converges pointwise to ϕ on the closed interval $[-c,c]$. We shall now use the General principle of uniform convergence. For every $m \in \mathbb{N}$, consider the partial sum

$$S_m(x) = \sum_{n=1}^{m} \frac{2(-1)^{n+1}}{n} \sin nx.$$

For every $m,k \in \mathbb{N}$ such that $m > k$, consider the difference

$$|S_m(x) - S_k(x)| = 2\left|\frac{\sin(k+1)x}{k+1} - \frac{\sin(k+2)x}{k+2} + \ldots \pm \frac{\sin mx}{m}\right|.$$

It is not difficult to check that

$$\begin{aligned}
&2\left(\frac{\sin(k+1)x}{k+1} - \frac{\sin(k+2)x}{k+2} + \ldots \pm \frac{\sin mx}{m}\right)\cos\frac{x}{2} \\
&= \left(\frac{\sin(k+\frac{3}{2})x}{k+1} + \frac{\sin(k+\frac{1}{2})x}{k+1}\right) - \left(\frac{\sin(k+\frac{5}{2})x}{k+2} + \frac{\sin(k+\frac{3}{2})x}{k+2}\right) \\
&\quad + \left(\frac{\sin(k+\frac{7}{2})x}{k+3} + \frac{\sin(k+\frac{5}{2})x}{k+3}\right) - \ldots \pm \left(\frac{\sin(m+\frac{1}{2})x}{m} + \frac{\sin(m-\frac{1}{2})x}{m}\right) \\
&= \frac{\sin(k+\frac{1}{2})x}{k+1} + \left(\frac{\sin(k+\frac{3}{2})x}{k+1} - \frac{\sin(k+\frac{3}{2})x}{k+2}\right) - \left(\frac{\sin(k+\frac{5}{2})x}{k+2} - \frac{\sin(k+\frac{5}{2})x}{k+3}\right) + \ldots \\
&\quad \mp \left(\frac{\sin(m-\frac{1}{2})x}{m-1} - \frac{\sin(m-\frac{1}{2})x}{m}\right) \pm \frac{\sin(m+\frac{1}{2})x}{m} \\
&= \frac{\sin(k+\frac{1}{2})x}{k+1} + \frac{\sin(k+\frac{3}{2})x}{(k+1)(k+2)} - \frac{\sin(k+\frac{5}{2})x}{(k+2)(k+3)} + \ldots \mp \frac{\sin(m-\frac{1}{2})x}{(m-1)m} \pm \frac{\sin(m+\frac{1}{2})x}{m},
\end{aligned}$$

so that

$$|S_m(x) - S_k(x)|\left|\cos\frac{x}{2}\right| \leqslant \frac{1}{k+1} + \frac{1}{(k+1)(k+2)} + \frac{1}{(k+2)(k+3)} + \ldots + \frac{1}{(m-1)m} + \frac{1}{m} = \frac{2}{k+1}.$$

For every $x \in [-c,c]$, we have $|\cos(x/2)| \geqslant \cos(c/2)$, so that

$$|S_m(x) - S_k(x)| \leqslant \frac{2}{k+1}\sec\frac{c}{2} < \epsilon, \quad \text{if } m > k > \frac{2}{\epsilon}\sec\frac{c}{2}.$$

The result now follows from the General principle of uniform convergence. ○

PROOF OF THEOREM 4.7. For simplicity, we extend the function f to a 2π-periodic function defined on $\mathbb{R}$ in the natural way, and do likewise for the function ϕ in Theorem 4.8. Consider the point d_1 of discontinuity of f. Let $j_1 = f(d_1+0) - f(d_1-0)$ denote the *jump* of the discontinuity. We shall now use the function ϕ to create a discontinuity at d_1 with jump $-j_1$. We do this by shifting and renormalizing ϕ. Note first that the function ϕ has a discontinuity at π with jump -2π. It follows that the function $\phi(x+\pi-d_1)$ has a discontinuity at d_1 with jump -2π, and so the function

$$\frac{j_1}{2\pi}\phi(x+\pi-d_1)$$

has a discontinuity at d_1 with jump $-j_1$. Hence the function

$$f(x) + \frac{j_1}{2\pi}\phi(x+\pi-d_1)$$

is continuous at d_1. We now repeat this process. For every $i = 2, \ldots, k$, let $j_i = f(d_i + 0) - f(d_i - 0)$ denote the jump of the discontinuity of f at d_i. Then the function

$$\frac{j_i}{2\pi}\phi(x + \pi - d_i)$$

has a discontinuity at d_i with jump $-j_i$. Hence the function

$$g(x) = f(x) + \frac{j_1}{2\pi}\phi(x + \pi - d_1) + \ldots + \frac{j_k}{2\pi}\phi(x + \pi - d_k)$$

is continuous at $d_1, \ldots, d_k$, and so continuous on the closed interval $[-\pi, \pi]$, with $g(-\pi) = g(\pi)$. It then follows from Theorem 4.6 that the Fourier series of g converges uniformly to g on $[-\pi, \pi]$, and hence also uniformly on every subinterval $[a, b]$ of $[-\pi, \pi]$. Suppose now that $[a, b]$ does not contain any of the points $d_1, \ldots, d_k$. Then it follows from Theorem 4.8 that the Fourier series of each of the functions $\phi(x + \pi - d_i)$ converges uniformly to $\phi(x + \pi - d_i)$ on $[a, b]$. Hence the Fourier series of f converges uniformly to f on $[a, b]$. $\bigcirc$

4.4. Parseval Identity

In this section, we answer the question of whether the infinite orthonormal system

$$\left\{\frac{1}{\sqrt{2}}, \sin x, \cos x, \sin 2x, \cos 2x, \sin 3x, \cos 3x, \ldots\right\}$$

in E, under the inner product given by

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,\mathrm{d}x, \quad \text{if } f, g \in E,$$

is sufficient to describe every function $f \in E$ as a Fourier series. It is shown earlier in Chapter 2 that the answer to this question lies in the idea of a closed system. In other words, we need

$$\lim_{m\to\infty} \|f - S_m\| = 0, \tag{4.10}$$

where for every $m \in \mathbb{N}$,

$$S_m(x) = \frac{a_0}{2} + \sum_{n=1}^{m}(a_n \cos nx + b_n \sin nx) \tag{4.11}$$

denotes the m-th partial sum of the Fourier series of f. More precisely, we establish the following result.

THEOREM 4.9. *For every $f \in E$ with Fourier series*

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{4.12}$$

we have

$$\lim_{m\to\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x) - S_m(x)|^2\,\mathrm{d}x = 0,$$

where for every $m \in \mathbb{N}$, the m-th partial sum $S_m(x)$ of the Fourier series is given by (4.11)*. In other words,* (4.10) *holds.*

In particular, it is shown in Theorem 2.13 that a closed system is equivalent to the Parseval identity being valid. The following result is then equivalent to Theorem 4.9.

THEOREM 4.10 (Parseval identity). *For every $f \in E$ with Fourier series* (4.12)*, we have*

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\,\mathrm{d}x = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty}\left(|a_n|^2 + |b_n|^2\right).$$

Our proof of Theorem 4.9 is split up into three steps. First of all, we establish Theorem 4.9 under some extra assumptions on the functions f. We then show that every function $f \in E$ can be approximated to some function that satisfies these extra assumptions. Finally, we deduce Theorem 4.9 in its generality. Corresponding to the first two steps, we establish the following two results.

THEOREM 4.11. *The conclusion of Theorem 4.9 holds for every $f \in E$ such that f is continuous on $[-\pi, \pi]$ with $f(-\pi) = f(\pi)$ and $f' \in E$.*

PROOF. It follows from Theorem 4.6 that the Fourier series of f converges uniformly to f on $[-\pi, \pi]$. Given any $\epsilon > 0$, there exists $N = N(\epsilon)$ such that

$$|f(x) - S_m(x)| < \tfrac{1}{2}\epsilon^{\frac{1}{2}}, \quad \text{if } m \geqslant N(\epsilon) \text{ and } x \in [-\pi, \pi].$$

It follows that for every $m \geqslant N(\epsilon)$, we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x) - S_m(x)|^2 \, \mathrm{d}x \leqslant \frac{\epsilon}{2}.$$

The result follows immediately. ○

THEOREM 4.12. *Suppose that $f \in E$. Then for every $\epsilon > 0$, there exists a function $g \in E$, continuous on $[-\pi, \pi]$ with $g(-\pi) = g(\pi)$ and $g' \in E$, such that $\|f - g\| < \epsilon$.*

We shall only indicate the ideas of the proof. The interested reader can make this formal without too great an effort.

SKETCH OF PROOF OF THEOREM 4.12. The idea here is first to approximate f by a step function h, and then to smooth the step function h to a continuous and piecewise linear function g. Note that $g' \in E$ if it is continuous and piecewise linear.

To approximate f to a step function h, note that f is Riemann integrable over $[-\pi, \pi]$. It follows that the integral of f over $[-\pi, \pi]$ can be approximated to any degree of accuracy by a Riemann sum. In other words, for every $\epsilon > 0$, we can find a step function h such that $\|f - h\| < \frac{1}{2}\epsilon$. Let this step function h be described by a dissection

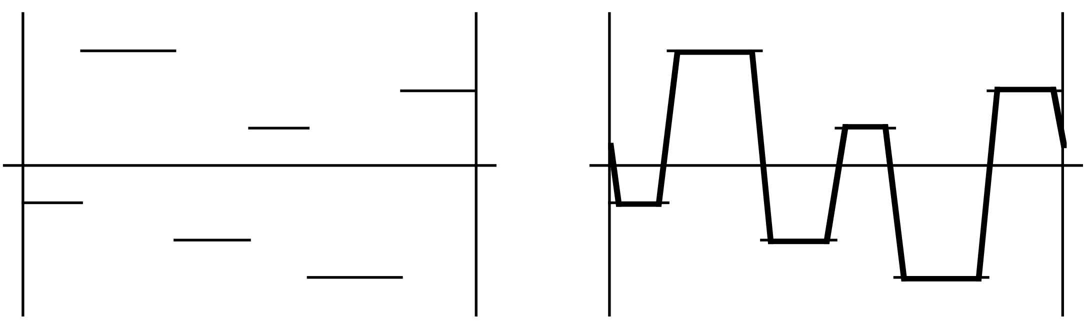
$$\Delta \; : \; -\pi = x_0 < x_1 < \ldots < x_k = \pi$$

of the interval $[-\pi, \pi]$, and by writing $h(x) = c_i$ whenever $x \in (x_{i-1}, x_i]$ for every $i = 1, \ldots, k$, with the convention that $h(x_0) = h(-\pi) = c_1$. Furthermore, the function f is clearly bounded in the interval $[-\pi, \pi]$, with $|f(x)| \leqslant M$ for every $x \in [-\pi, \pi]$, say. Then we can further require that the function h satisfies the condition that $|h(x)| \leqslant M$ for every $x \in [-\pi, \pi]$.

To smooth h to a continuous and piecewise linear function g, we choose a suitable $\delta > 0$, sufficiently small so that 2δ is smaller than the length of each of the subintervals of the dissection Δ. More precisely, we require that $2\delta < x_i - x_{i-1}$ for every $i = 1, \ldots, k$. Then each subinterval $[x_{i-1}, x_i]$ can now be split into three subintervals

$$[x_{i-1}, x_i] = [x_{i-1}, x_{i-1} + \delta] \cup [x_{i-1} + \delta, x_i - \delta] \cup [x_i - \delta, x_i].$$

We now define $g(x) = c_i$ whenever $x \in [x_{i-1} + \delta, x_i - \delta]$ for every $i = 1, \ldots, k$, so that $g(x) = h(x)$ for these values of x. Note that if δ is very small, then $g(x) = h(x)$ for a substantial proportion of the interval $[-\pi, \pi]$. We then join up these pieces of horizontal segments by linear segments. Below, the picture on the left shows an example of the graph of a step function h, while the picture on the right shows that the graph of continuous and piecewise linear function g in bold lines superimposed on the graph of h.



More precisely, for every $i = 1, \ldots, k-1$, we write

$$g(x) = \frac{c_{i+1} + c_i}{2} + \frac{c_{i+1} - c_i}{2\delta}(x - x_i), \quad \text{if } x \in [x_i - \delta, x_i + \delta].$$

Furthermore, we write

$$g(x)=\begin{cases}\dfrac{c_1+c_n}{2}+\dfrac{c_1-c_n}{2\delta}(x+\pi), & \text{if } x\in[-\pi,-\pi+\delta],\\ \dfrac{c_1+c_n}{2}+\dfrac{c_1-c_n}{2\delta}(x-\pi), & \text{if } x\in[\pi-\delta,\pi].\end{cases}$$

Then

$$\begin{aligned}\|h-g\|^2&=\frac{1}{\pi}\int_{-\pi}^{\pi}|h(x)-g(x)|^2\,\mathrm{d}x=\frac{1}{\pi}\sum_{i=1}^{k}\left(\int_{x_{i-1}}^{x_{i-1}+\delta}|h(x)-g(x)|^2\,\mathrm{d}x+\int_{x_i-\delta}^{x_i}|h(x)-g(x)|^2\,\mathrm{d}x\right)\\ &\leqslant\frac{1}{\pi}\sum_{i=1}^{k}\left(\int_{x_{i-1}}^{x_{i-1}+\delta}4M^2\,\mathrm{d}x+\int_{x_i-\delta}^{x_i}4M^2\,\mathrm{d}x\right)=\frac{8M^2\delta k}{\pi}.\end{aligned}$$

Note that M depends only on f, while k depends only on h and ϵ and hence only on f and ϵ. It follows that by choosing $\delta>0$ sufficiently small, we can ensure that $\|h-g\|<\frac{1}{2}\epsilon$. It follows immediately from the Triangle inequality that $\|f-g\|\leqslant\|f-h\|+\|h-g\|<\epsilon$. ◯

Proof of Theorem 4.9. Suppose that $f\in E$. Given any $\epsilon>0$, we use Theorem 4.12 to convlude that there exists a function $g\in E$, continuous on $[-\pi,\pi]$ with $g(-\pi)=g(\pi)$ and $g'\in E$, such that $\|f-g\|<\frac{1}{2}\epsilon$. For every $m\in\mathbb{N}$, let $S_m(x)$ and $T_m(x)$ denote respectively the m-th partial sums of the Fourier series of f and g. Then it follows from Theorem 4.11 that there exists $N=N(\epsilon)$ such that

$$\|g-T_m\|<\tfrac{1}{2}\epsilon,\quad \text{if } m\geqslant N(\epsilon).$$

Using the Triangle inequality, we conclude that

$$\|f-T_m\|\leqslant\|f-g\|+\|g-T_m\|<\epsilon,\quad \text{if } m\geqslant N(\epsilon).$$

On the other hand, using the special case of Theorem 2.9(iii) as applied to Fourier series, we conclude that S_m, being the orthogonal projection of f on the subspace

$$\operatorname{span}\left\{\frac{1}{\sqrt{2}},\sin x,\cos x,\sin 2x,\cos 2x,\sin 3x,\cos 3x,\ldots,\sin mx,\cos mx\right\}$$

of E, satisfies the inequality $\|f-S_m\|\leqslant\|f-T_m\|$. It follows that

$$\|f-S_m\|<\epsilon,\quad \text{if } m\geqslant N(\epsilon).$$

This completes the proof. ◯

Example 4.4.1. Consider the function $f:[-\pi,\pi]\to\mathbb{C}$, given by $f(x)=x^2$ for every $x\in[-\pi,\pi]$. It is shown in Example 3.1.4 that we have the Fourier series

$$x^2\sim\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx.$$

Using the Parseval identity as given in Theorem 4.10, we conclude that

$$\frac{1}{\pi}\int_{-\pi}^{\pi}x^4\,\mathrm{d}x=\frac{2\pi^4}{9}+16\sum_{n=1}^{\infty}\frac{1}{n^4}.$$

If we evaluate the integral on the left hand side, we conclude that

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}.$$

Remark. It is easy to deduce that the sequence

$$\{1,\mathrm{e}^{\mathrm{i}x},\mathrm{e}^{-\mathrm{i}x},\mathrm{e}^{2\mathrm{i}x},\mathrm{e}^{-2\mathrm{i}x},\mathrm{e}^{3\mathrm{i}x},\mathrm{e}^{-3\mathrm{i}x},\ldots\}$$

in E forms a closed infinite orthonormal system under the inner product given by

$$\langle f,g\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\overline{g(x)}\,\mathrm{d}x,\quad \text{if } f,g\in E.$$

The corresponding Parseval identity is given by

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,\mathrm{d}x = \sum_{n=-\infty}^{\infty}|c_n|^2.$$

On the other hand, Theorem 2.14 applied to the case of Fourier series gives the following result.

THEOREM 4.13 (Generalized Parseval identity). *Suppose that $f, g \in E$ have Fourier series*

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad \textit{and} \quad g(x) \sim \frac{c_0}{2} + \sum_{n=1}^{\infty}(c_n \cos nx + d_n \sin nx)$$

respectively. Then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,\mathrm{d}x = \frac{a_0\overline{c_0}}{2} + \sum_{n=1}^{\infty}(a_n\overline{c_n} + b_n\overline{d_n}).$$

We conclude this chapter by establishing the uniqueness of Fourier series.

THEOREM 4.14. *Suppose that $f, g \in E$ have identical Fourier series. Then $f(x) = g(x)$ for every $x \in [-\pi, \pi]$ with at most finitely many exceptions.*

PROOF. Note that $f - g \in E$ has all Fourier coefficients equal to zero. The Parseval identity then gives $\|f - g\| = 0$. ◯

Problems for Chapter 4

1. Consider the function $f : [-\pi, \pi] \to \mathbb{C}$, given by $f(x) = 1 - x^2$ for every $x \in [-\pi, \pi]$.
   - (i) Calculate the coefficients of the trigonometric Fourier series of the function.
   - (ii) To what values does the Fourier series converge at the points $x = 275\pi$ and $x = -360\pi$?
2. Consider the function $f : [-\pi, \pi] \to \mathbb{C}$, given by $f(x) = x/\pi - [x/\pi] - \frac{1}{2}$ for every $x \in [-\pi, \pi]$.
   - (i) Calculate the coefficients of the trigonometric Fourier series of the function.
     [*Remark*: Here $[z]$ denotes the greatest integer not exceeding z.]
   - (ii) To what values does the Fourier series converge at the points $x = 0$ and $x = -\pi/2$?
3. Suppose that

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is the Fourier series of the function $f(x) = \mathrm{e}^{px}$ in the interval $[-\pi, \pi]$, where p is a fixed non-zero real number. Determine the values of

$$\sum_{n=0}^{\infty} a_n \quad \text{and} \quad \sum_{n=0}^{\infty} (-1)^n a_n.$$

4. Consider the function $f : [-\pi, \pi] \to \mathbb{C}$, given by

$$f(x) = \begin{cases} \sin x, & \text{if } -\pi \leqslant x < 0, \\ \cos x, & \text{if } 0 \leqslant x \leqslant \pi. \end{cases}$$

   - (i) Draw the graph of the Fourier series of f in the interval $[-3\pi, 3\pi]$ without finding the Fourier series.
   - (ii) To what values does the Fourier series converge at the points $x = 0$ and $x = \pm\pi$?

5. Suppose that the function $f \in E$ satisfies the conditions of Dirichlet's theorem. Evaluate each of the following limits:
   - (i) $\displaystyle\lim_{n\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, \mathrm{d}x$
   - (ii) $\displaystyle\lim_{n\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(n + \tfrac{1}{2})x \, \mathrm{d}x$
   - (iii) $\displaystyle\lim_{n\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} x^{-1} f(x) \sin(n + \tfrac{1}{2})x \, \mathrm{d}x$
6. Consider the function $f : [-\pi, \pi] \to \mathbb{C}$, given by

$$f(x) = \begin{cases} 0, & \text{if } -\pi \leqslant x < 0, \\ \sin x, & \text{if } 0 \leqslant x \leqslant \pi. \end{cases}$$

   - (i) Find the Fourier series of f.
   - (ii) Show that $\displaystyle\frac{\pi - 2}{4} = \frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \frac{1}{7 \cdot 9} + \ldots.$
   - (iii) Show that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{4k^2 - 1} = \frac{1}{2}.$

7. For every $m \in \mathbb{N}$, let $\displaystyle D_m(t) = \frac{1}{2} + \sum_{n=1}^{m} \cos nt.$
   - (i) Determine $\displaystyle\int_{-\pi}^{\pi} D_m(t) \sin 2t \, \mathrm{d}t$ for every $m \in \mathbb{N}$.
   - (ii) Evaluate $\displaystyle\int_{-\pi}^{\pi} |D_{20}(t)|^2 \, \mathrm{d}t.$
   - (iii) Evaluate $\displaystyle\lim_{m\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} D_m(t) g(t) \, \mathrm{d}t$, where $g(t) = t^{-1} \sin \frac{1}{2}t$ for every non-zero $t \in [-\pi, \pi]$.

8. Determine values of the constants A and B such that the function $f : [-\pi, \pi] \to \mathbb{C}$, defined by

$$f(x) = \begin{cases} Ax + B, & \text{if } -\pi \leqslant x < 0, \\ \cos x, & \text{if } 0 \leqslant x \leqslant \pi, \end{cases}$$

has a Fourier series that converges uniformly to f in the interval $[-\pi, \pi]$.

9. Suppose that $a \in \mathbb{R} \setminus \mathbb{Z}$.

(i) Prove that for every $x \in [-\pi, \pi]$, we have

$$\cos ax = \frac{\sin \pi a}{\pi a} + \sum_{n=1}^{\infty} (-1)^n \frac{2a \sin \pi a}{\pi(a^2 - n^2)} \cos nx.$$

(ii) Deduce that

$$\cot \pi a = \frac{1}{\pi} \left(\frac{1}{a} - \sum_{n=1}^{\infty} \frac{2a}{n^2 - a^2} \right).$$

10. Suppose that a function $f \in E$ has Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where for every $n \in \mathbb{N}$, the coefficients a_n and b_n satisfy

$$|a_n| \leqslant \frac{1}{n^2} \quad \text{and} \quad |b_n| \leqslant \frac{1}{n^2}.$$

Show that f may be considered to be continuous in $[-\pi, \pi]$ with $f(-\pi) = f(\pi)$, and that the Fourier series of f converges to f uniformly in the interval $[-\pi, \pi]$.

11. A trigonometric polynomial is a finite linear combination of functions in the set

$$\{1, \cos x, \sin x, \cos 2x, \sin 2x, \cos 3x, \sin 3x, \ldots\},$$

with coefficients in $\mathbb{C}$. For instance, partial sums of a Fourier series are trigonometric polynomials. Suppose that $f \in E$ is continuous on $[-\pi, \pi]$ and satisfies $f(-\pi) = f(\pi)$. Prove the Weierstrass approximation theorem, that f can be approximated by trigonometric polynomials uniformly in $[-\pi, \pi]$. In other words, show that given any $\epsilon > 0$, there exists a trigonometric polynomial T such that $|f(x) - T(x)| < \epsilon$ for every $x \in [-\pi, \pi]$.

12. Consider the function $f : [-\pi, \pi] \to \mathbb{C}$, given by

$$f(x) = \begin{cases} \sin 2x, & \text{if } -\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Find the Fourier series of f.
(ii) Find the Fourier series of f'.
(iii) To what values does the Fourier series of f' converge at the points $x = \pm\frac{1}{2}\pi$?
(iv) Evaluate the sums

$$\sum_{k=1}^{\infty} \frac{(-1)^k (2k-1)}{(2k-3)(2k+1)} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{(2k-3)^2 (2k+1)^2}.$$

13. Evaluate each of the following integrals:

(i) $$\int_{-\pi}^{\pi} \left| \sum_{n=1}^{\infty} \frac{\mathrm{e}^{inx}}{n^2} \right|^2 \mathrm{d}x$$

(ii) $$\int_{-\pi}^{\pi} \left| 1 + \sum_{n=1}^{m} (\cos nx - \sin nx) \right|^2 \mathrm{d}x$$

14. Suppose that the 2π-periodic function $f \in E$ has Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

For every $x \in [-\pi, \pi]$, let $g(x) = f(x + \pi) - f(x)$. Evaluate the integral

$$\int_{-\pi}^{\pi} |g(x)|^2 \, \mathrm{d}x$$

in terms of the Fourier coefficients of f.

15. Consider the function e^{-x} in the interval $[-\pi, \pi]$, with Fourier series

$$\mathrm{e}^{-x} \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

By first writing $\mathrm{e}^{-x} = g(x) + h(x)$, where $g(x)$ is an even function and $h(x)$ is an odd function, evaluate the quantities

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 \quad \text{and} \quad \sum_{n=1}^{\infty} b_n^2.$$