FIRST YEAR CALCULUS

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Chapter 3

INTRODUCTION TO DERIVATIVES

3.1. Introduction

We begin by looking at a simple example. Suppose that a car is travelling along a road. For 10 hours, its distance from the point of origin is noted at hourly intervals and recorded. The table below shows its distance x in kilometres from the point of origin against time t in hours:

t	0	1	2	3	4	5	6	7	8	9	10
x	0	50	120	190	290	350	470	560	620	690	750

We can denote by $s(t)$ the distance of the car from the point of origin after time t, so that $s(3) = 190$ and $s(8) = 620$, for example. Then the average speed of the car between the 3-hour mark and the 8-hour mark will be given by

$$\frac{\text{change in distance over the time interval}}{\text{length of the time interval}} = \frac{s(8) - s(3)}{8 - 3} = \frac{620 - 190}{8 - 3} = 86$$

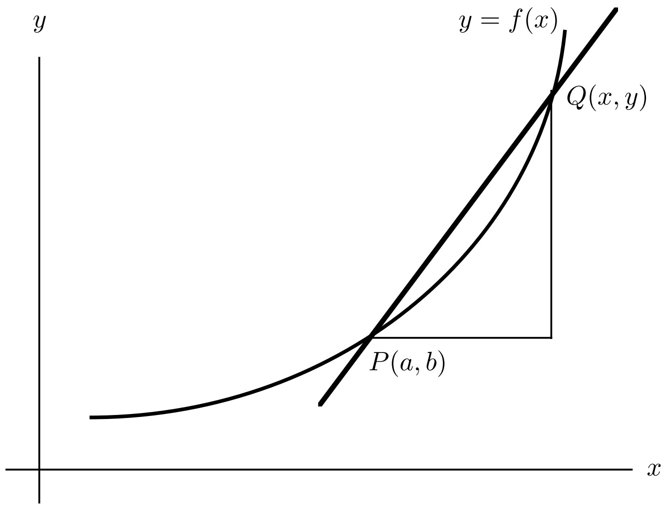
kilometres per hour. Suppose next that we wish to find the actual speed of the car at the 3-hour mark. Then the table above is not of much use. However, if more precise information of the position of the car is available at all time, then perhaps the following strategy may be useful. We take the position $s(3)$ of the car at the 3-hour mark. Now add a small time interval Δt, and find out the position $s(3 + \Delta t)$ of the car after $3 + \Delta t$ hours. Then we calculate the average speed

$$\frac{s(3 + \Delta t) - s(3)}{\Delta t}$$

of the car over this small time interval. If Δt is very small, then this average should be roughly the speed of the car at the 3-hour mark. We are therefore looking at some quantity, if it exists at all, like

$$\lim_{\Delta t \to 0} \frac{s(3 + \Delta t) - s(3)}{\Delta t}.$$

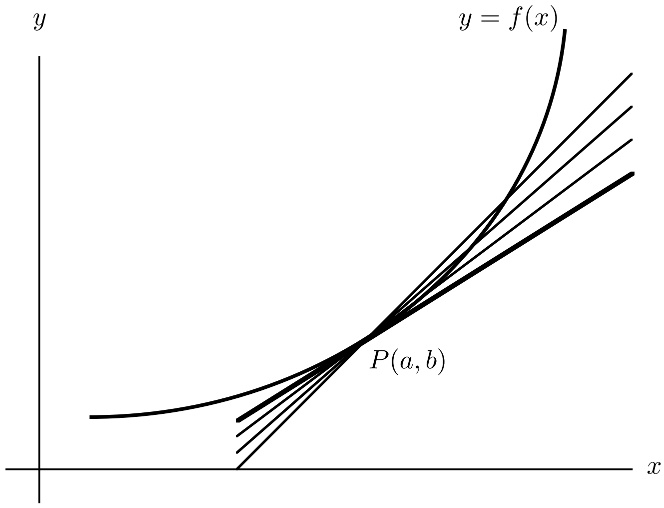
Consider the graph of a function $y = f(x)$. Suppose that $P(a, b)$ is a point on the curve $y = f(x)$. Consider now another point $Q(x, y)$ on the curve close to the point $P(a, b)$. We draw the line joining the points $P(a, b)$ and $Q(x, y)$, and obtain the picture below.



Clearly the slope of this line is equal to

$$\frac{y-b}{x-a} = \frac{f(x)-f(a)}{x-a}.$$

Now let us keep the point $P(a, b)$ fixed, and move the point $Q(x, y)$ along the curve towards the point P. Eventually the line PQ becomes the tangent to the curve $y = f(x)$ at the point $P(a, b)$, as shown in the picture below.

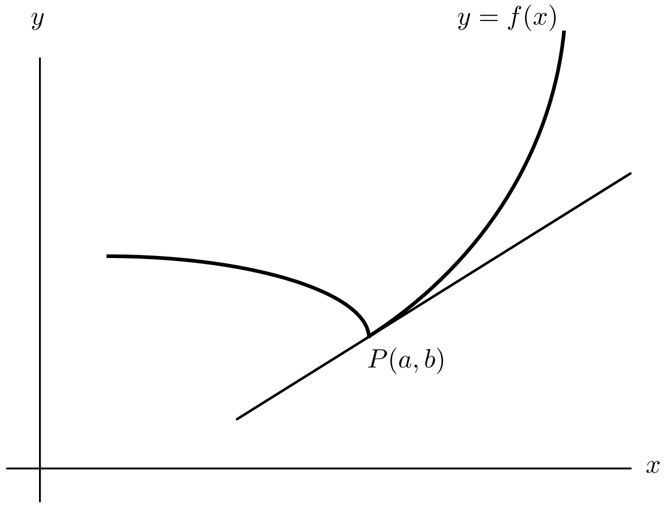


We are interested in the slope of this tangent line. Its value is called the derivative of the function $y = f(x)$ at the point $x = a$, and denoted by

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a} \qquad \text{or} \qquad f'(a).$$

In this case, we say that the function $y = f(x)$ is differentiable at the point $x = a$.

Remark. Sometimes, when we move the point $Q(x, y)$ along the curve $y = f(x)$ towards the point $P(a, b)$, the line PQ does not become the tangent to the curve $y = f(x)$ at the point $P(a, b)$. In this case, we say that the function $y = f(x)$ is not differentiable at the point $x = a$. An example of such a situation is given in the picture below.



Note that in this case, the curve $y = f(x)$ makes an abrupt turn at the point $P(a, b)$.

In this chapter, we assume that the reader has some idea of the notion of a limit of a function $f(x)$ as $x \to a$. In particular, we assume that the reader takes on trust the following result. The three parts are respectively called the sum, product and quotient rules for limits.

ARITHMETIC OF LIMITS. *Suppose that the functions $f(x) \to L$ and $g(x) \to M$ as $x \to a$. Then*
(a) $f(x) + g(x) \to L + M$ as $x \to a$;
(b) $f(x)g(x) \to LM$ as $x \to a$; and
(c) if $M \neq 0$, then $f(x)/g(x) \to L/M$ as $x \to a$.

In the remainder of this first section, we recall some well known facts concerning derivatives. The advanced reader may choose instead to proceed immediately to the next section. Indeed, we do not prove any statements in this chapter, as we have chosen not to develop the theory of limits at this point. The proofs of these statements will be given in Chapter 8. Here we begin by looking at a concrete example.

EXAMPLE 3.1.1. Consider the function $f(x) = x^2$. The reader should try to draw the graph of this function, and follow the description below. Let us consider the point $(2, 4)$ on the curve, and denote this point by P. Suppose that we wish to calculate the slope of the tangent to the curve at P. We may do the following. Let Q denote an arbitrary point (x, x^2), where x is close to 2. Then the line through P and Q has slope

$$\frac{x^2 - 4}{x - 2}.$$

Suppose now that we move the point Q towards the point P along the curve. Then as Q approaches P, this line through P and Q approaches the tangent at P, while its slope approaches the value

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}.$$

This value must then be the slope of the tangent at P.

We can obviously repeat the same argument with any arbitrary function $f(x)$, and investigate whether there is a tangent at the point $(a, f(a))$.

DEFINITION. We say that a function $f(x)$ is differentiable at $x = a$ if the limit

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} \tag{1}$$

exists. In this case, the value (1) is denoted by $f'(a)$ and is called the derivative of $f(x)$ at $x = a$.

EXAMPLE 3.1.2. Consider the function $f(x) = c$, where $c \in \mathbb{R}$ is a constant. For every $a \in \mathbb{R}$, we have

$$\frac{f(x) - f(a)}{x - a} = 0 \to 0$$

as $x \to a$. It follows that $f'(a) = 0$ for every $a \in \mathbb{R}$.

EXAMPLE 3.1.3. Consider the function $f(x) = x$. For every $a \in \mathbb{R}$, we have

$$\frac{f(x) - f(a)}{x - a} = 1 \to 1$$

as $x \to a$. It follows that $f'(a) = 1$ for every $a \in \mathbb{R}$.

EXAMPLE 3.1.4. Consider the function $f(x) = x^n$, where $n \geq 2$ is an integer. For every $a \in \mathbb{R}$, we have

$$\frac{f(x) - f(a)}{x - a} = \frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + x^2a^{n-3} + xa^{n-2} + a^{n-1} \to na^{n-1}$$

as $x \to a$. It follows that $f'(a) = na^{n-1}$ for every $a \in \mathbb{R}$.

EXAMPLE 3.1.5. Consider the function $f(x) = \sqrt{x}$. For every positive $a \in \mathbb{R}$, we have

$$\frac{f(x) - f(a)}{x - a} = \frac{\sqrt{x} - \sqrt{a}}{x - a} = \frac{\sqrt{x} - \sqrt{a}}{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})} = \frac{1}{\sqrt{x} + \sqrt{a}} \to \frac{1}{2\sqrt{a}}$$

as $x \to a$. It follows that $f'(a) = 1/2\sqrt{a}$ for every positive $a \in \mathbb{R}$.

EXAMPLE 3.1.6. Consider the function $f(x) = \sin x$. For every $a \in \mathbb{R}$, we have

$$\frac{f(x) - f(a)}{x - a} = \frac{\sin x - \sin a}{x - a} = \frac{2\cos\frac{1}{2}(x + a)\sin\frac{1}{2}(x - a)}{x - a} = \frac{\sin\frac{1}{2}(x - a)}{\frac{1}{2}(x - a)}\cos\frac{1}{2}(x + a) \to \cos a$$

as $x \to a$. It follows that $f'(a) = \cos a$ for every $a \in \mathbb{R}$.

EXAMPLE 3.1.7. Consider the function $f(x) = \cos x$. For every $a \in \mathbb{R}$, we have

$$\frac{f(x) - f(a)}{x - a} = \frac{\cos x - \cos a}{x - a} = -\frac{2\sin\frac{1}{2}(x + a)\sin\frac{1}{2}(x - a)}{x - a} = -\frac{\sin\frac{1}{2}(x - a)}{\frac{1}{2}(x - a)}\sin\frac{1}{2}(x + a) \to -\sin a$$

as $x \to a$. It follows that $f'(a) = -\sin a$ for every $a \in \mathbb{R}$.

Example 3.1.4 above raises the question of determining derivatives of functions of the type $f(x) = x^n$, where n is a real number, not necessarily a positive integer. We state, without proof, the following important result.

PROPOSITION 3A. *Suppose that $n \in \mathbb{R}$ is fixed and non-zero. Then for the function $f(x) = x^n$, we have $f'(a) = na^{n-1}$ for every $a \in \mathbb{R}$ for which a^{n-1} is defined.*

EXAMPLE 3.1.8. Consider the function $f(x) = x^{35/36}$. We have

$$f'(a) = \tfrac{35}{36} a^{-1/36}$$

for every positive $a \in \mathbb{R}$.

We have the sum, product and quotient rules for derivatives. In Chapter 8, we shall establish the following result.

PROPOSITION 3B. *Suppose that the functions $f(x)$ and $g(x)$ are differentiable at $x = a$. Then*
(a) $f(x) + g(x)$ is differentiable at $x = a$, with $(f + g)'(a) = f'(a) + g'(a)$;
(b) $f(x)g(x)$ is differentiable at $x = a$, with $(fg)'(a) = f(a)g'(a) + f'(a)g(a)$; and
(c) if $g(a) \neq 0$, then $f(x)/g(x)$ is differentiable at $x = a$, with $\displaystyle\left(\frac{f}{g}\right)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{g^2(a)}$.

EXAMPLE 3.1.9. Consider the function $f(x) = \tan x$. We know that

$$\tan x = \frac{\sin x}{\cos x}.$$

It follows that for every $a \in \mathbb{R}$ such that $\cos a \neq 0$, we have, by the quotient rule, that

$$f'(a) = \frac{\cos^2 a + \sin^2 a}{\cos^2 a} = \frac{1}{\cos^2 a} = \sec^2 a.$$

EXAMPLE 3.1.10. Consider the function $f(x) = \csc x$. We know that

$$\csc x = \frac{1}{\sin x}.$$

It follows that for every $a \in \mathbb{R}$ such that $\sin a \neq 0$, we have, by the quotient rule, that

$$f'(a) = \frac{0 - \cos a}{\sin^2 a} = -\cot a \csc a.$$

EXAMPLE 3.1.11. Consider the function

$$f(x) = \frac{x^3 \sin x}{x^2 + 3}.$$

We can write $f(x) = g(x)/h(x)$, where $g(x) = x^3 \sin x$ and $h(x) = x^2 + 3$. For every $a \in \mathbb{R}$, we have $g'(a) = a^3 \cos a + 3a^2 \sin a$ and $h'(a) = 2a$. It follows that

$$f'(a) = \frac{h(a)g'(a) - g(a)h'(a)}{h^2(a)} = \frac{(a^2 + 3)(a^3 \cos a + 3a^2 \sin a) - 2a^4 \sin a}{(a^2 + 3)^2}.$$

From now on, we shall slightly abuse our notation, and simply refer to $f'(x)$ as the derivative of the function $f(x)$. We shall further write

$$y = f(x) \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = f'(x).$$

It follows, for example, that if we write

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{\sin x}\right) = \frac{\sin x - x\cos x}{\sin^2 x},$$

then we mean that we are considering the function $f(x) = x/\sin x$, and that for every $a \in \mathbb{R}$ for which $\sin a \neq 0$, we have $f'(a) = (\sin a - a\cos a)/\sin^2 a$.

An important technique in differentiation is through the use of composite functions. We begin by looking at an example.

EXAMPLE 3.1.12. Let $y = (x^3+1)^2$. To calculate the derivative $\mathrm{d}y/\mathrm{d}x$, note that we can first of all write $y = x^6 + 2x^3 + 1$, and then differentiate to obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^5 + 6x^2 = 6x^2(x^3+1).$$

Let us look at this in a different way. We can write $y = u^2$, where $u = x^3 + 1$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}u} = 2u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2.$$

Note that

$$\frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = 6ux^2 = 6x^2(x^3+1).$$

We therefore have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Perhaps this is a coincidence. However, let us investigate further. If we write $u = f(x) = x^3 + 1$ and $y = g(u) = u^2$, then $(g \circ f)(x) = g(f(x)) = g(x^3+1) = (x^3+1)^2$. It follows that our original function is really a composition of two functions. As we vary x, the value $u = f(x)$ changes at the rate of $\mathrm{d}u/\mathrm{d}x$. This change in the value of $u = f(x)$ in turn causes a change in the value of $y = g(u)$ at the rate of $\mathrm{d}y/\mathrm{d}u$. It is therefore not unreasonable to expect the change in x causes a change in y at the rate $(\mathrm{d}y/\mathrm{d}u)(\mathrm{d}u/\mathrm{d}x)$.

Indeed, this is the case, and the following result is known as the Chain rule for differentiation which we shall prove in Chapter 8.

PROPOSITION 3C. *Suppose that y is a differentiable function of u, and that u is a differentiable function of x. Then y is a differentiable function of x. Furthermore, we have*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

EXAMPLE 3.1.13. Let $y = \sqrt{(1+x^2)^5}$. Then we can write $y = u^{5/2}$, where $u = 1 + x^2$, so that

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{5}{2}u^{3/2} \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x.$$

It follows that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = 5xu^{3/2} = 5x(1+x^2)^{3/2}.$$

EXAMPLE 3.1.14. Let $y = \sin(x^5 + 3x)$. Then we can write $y = \sin u$, where $u = x^5 + 3x$, so that

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \cos u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 5x^4 + 3.$$

It follows that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = (5x^4 + 3)\cos u = (5x^4 + 3)\cos(x^5 + 3x).$$

EXAMPLE 3.1.15. Let $y = \sin(x^2 + 1)^{1/2}$. Then we can write $y = \sin u$, where $u = v^{1/2}$ and $v = x^2 + 1$, so that

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \cos u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}v} = \frac{1}{2v^{1/2}} \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 2x.$$

It follows that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{x\cos u}{v^{1/2}} = \frac{x\cos(x^2 + 1)^{1/2}}{(x^2 + 1)^{1/2}}.$$

3.2. Stationary Points and Second Derivatives

We have indicated earlier that the derivative $f'(a)$ of a function $f(x)$ at a point $x = a$ may be interpreted as the slope of the tangent at the point $(a, f(a))$ to the curve representing the function $f(x)$. It follows that if $f'(a) > 0$, then the function $f(x)$ must be increasing near $x = a$, whereas if $f'(a) < 0$, then the function $f(x)$ must be decreasing near $x = a$. We now want to find a suitable interpretation for the case $f'(a) = 0$. A fair guess would be to suspect that the function is neither increasing nor decreasing. However, this is not quite correct. To begin a proper study of this question, we need a few definitions.

DEFINITION.
(1) A function $f(x)$ is said to have a local maximum at $x = a$ if there is an open interval I containing the real number a and such that $f(x) \le f(a)$ for every $x \in I$.
(2) A function $f(x)$ is said to have a local minimum at $x = a$ if there is an open interval I containing the real number a and such that $f(x) \ge f(a)$ for every $x \in I$.
(3) A function $f(x)$ is said to have a stationary point at $x = a$ if $f'(a) = 0$.

EXAMPLE 3.2.1. Try to make a rough sketch of the graph of the function $f(x) = x^2$. Since $f'(x) = 2x$ for every $x \in \mathbb{R}$, the only stationary point is at $x = 0$. On the other hand, note that for every $x \ne 0$, we have $f(x) = x^2 > 0 = f(0)$. It follows that there is a local minimum at $x = 0$.

EXAMPLE 3.2.2. Try to make a rough sketch of the graph of the function $f(x) = x^3$. Since $f'(x) = 3x^2$ for every $x \in \mathbb{R}$, the only stationary point is at $x = 0$. On the other hand, note that for every $x < 0$, we have $f(x) = x^3 < 0 = f(0)$, whereas for every $x > 0$, we have $f(x) = x^3 > 0 = f(0)$. It follows that $x = 0$ does not represent a local minimum or a local maximum.

To detect a local minimum or local maximum, the (first) derivative of a function $f(x)$ is a useful tool. After all, if a continuous function increases before a point $x = a$ and decreases after $x = a$, it is reasonable to accept the point $x = a$ as representing a local maximum. On the other hand, if a continuous function decreases before a point $x = a$ and increases after $x = a$, it is reasonable to accept the point $x = a$ as representing a local minimum. Note that we have not defined what a continuous function is, but to understand the previous two sentences, it is enough to note that a function defined on any interval is continuous in that interval if we can draw the graph of the function on that interval without lifting the pen from the paper.

Indeed, we shall establish the following result in Chapter 8.

PROPOSITION 3D. *Suppose that I is an open interval containing a. Suppose further that a function $f(x)$ is continuous in I, and differentiable at every $x \in I$, except possibly at $x = a$.*
(a) If $f'(x) > 0$ for every $x < a$ in I and $f'(x) < 0$ for every $x > a$ in I, then the function $f(x)$ has a local maximum at $x = a$.
(b) If $f'(x) < 0$ for every $x < a$ in I and $f'(x) > 0$ for every $x > a$ in I, then the function $f(x)$ has a local minimum at $x = a$.

EXAMPLE 3.2.3. Let us return to the function $f(x) = x^2$. Since $f'(x) = 2x$ for every $x \in \mathbb{R}$, it is clear that $f'(x) < 0$ for every $x < 0$ and $f'(x) > 0$ for every $x > 0$. It follows that there is a local minimum at $x = 0$.

EXAMPLE 3.2.4. Consider the function $f(x) = 2x^3 - 9x^2 + 12x - 5$. Since

$$f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 2)$$

for every $x \in \mathbb{R}$, it is clear that the only stationary points are at $x = 1$ and $x = 2$. To determine whether either of these represents a local maximum or a local minimum, we study the function $f'(x)$ more closely. It is easy to see that

$$f'(x) \begin{cases} > 0 & \text{if } x \in (0,1), \\ < 0 & \text{if } x \in (1,2), \\ > 0 & \text{if } x \in (2,3). \end{cases}$$

It follows that $f(x)$ has a local maximum at $x = 1$ and a local minimum at $x = 2$.

EXAMPLE 3.2.5. Try to make a rough sketch of the graph of the function $f(x) = 1 - |x|$. It is not difficult to see that $f(x)$ is continuous everywhere. Furthermore, we have

$$f(x) = \begin{cases} 1 + x & \text{if } x < 0, \\ 1 - x & \text{if } x > 0, \end{cases}$$

so that

$$f'(x) = \begin{cases} 1 & \text{if } x < 0, \\ -1 & \text{if } x > 0. \end{cases}$$

It follows that $f(x)$ has a local maximum at $x = 0$. Note also that $f(x)$ is not differentiable at $x = 0$.

If the first derivative measures the rate of change of a function, then the second derivative measures the rate of change of the first derivative. Since the first derivative represents the slope of the tangent to the curve, it follows that the second derivative measures the rate of change of this slope.

Suppose now that we have a function $f(x)$ differentiable in an open interval containing a. Imagine that $f(x)$ has a local maximum at $x = a$. Then we cannot have $f'(a) > 0$, otherwise $f(x)$ is increasing at $x = a$. Also, we cannot have $f'(a) < 0$, otherwise $f(x)$ is decreasing at $x = a$. It follows that we must have $f'(a) = 0$. This means that the tangent to the curve at $x = a$ is horizontal. If we move a little to the right from $x = a$, then clearly $f(x)$ decreases, so that the tangent to the curve now has a negative slope. On the other hand, if we move a little to the left from $x = a$, then clearly $f(x)$ also decreases, so that the tangent to the curve now has a positive slope. It follows that if we move from a little to the left of $x = a$ to a little to the right of $x = a$, the slope of the tangent changes from positive to negative. Hence the slope of the tangent is decreasing. This means that the second derivative must be negative.

The above heuristics can be summarized by the following result on stationary points and second derivatives which we shall establish formally in Chapter 8.

PROPOSITION 3E. *Suppose that I is an open interval containing a real number a. Suppose further that the function $f(x)$ is differentiable at every $x \in I$, and that $f'(a) = 0$.*
(a) If $f''(a) < 0$, then the function $f(x)$ has a local maximum at $x = a$.
(b) If $f''(a) > 0$, then the function $f(x)$ has a local minimum at $x = a$.

EXAMPLE 3.2.6. Let us return to the function $f(x) = x^2$. Since $f'(x) = 2x$ for every $x \in \mathbb{R}$, it is clear that $x = 0$ is the only stationary point. On the other hand, we have $f''(x) = 2$ for every $x \in \mathbb{R}$, so that $f''(0) > 0$. It follows that $f(x)$ has a local minimum at $x = 0$.

EXAMPLE 3.2.7. Consider the function $f(x) = 2x^3 - 9x^2 + 12x - 5$. Since

$$f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 2)$$

for every $x \in \mathbb{R}$, it is clear that the only stationary points are at $x = 1$ and $x = 2$. On the other hand, we have $f''(x) = 12x - 18$ for every $x \in \mathbb{R}$, so that $f''(1) < 0$ and $f''(2) > 0$. It follows that $f(x)$ has a local maximum at $x = 1$ and a local minimum at $x = 2$.

DEFINITION. We say that a function $f(x)$ has a point of inflection at $x = a$ if $f''(a) = 0$.

EXAMPLE 3.2.8. Recall our discussion of the function $f(x) = x^3$ in Example 3.2.2. Since $f'(x) = 3x^2$ and $f''(x) = 6x$ for every $x \in \mathbb{R}$, the only stationary point is at $x = 0$. Furthermore, we have $f'(0) = 0$ and $f''(0) = 0$. We have shown earlier that $x = 0$ does not represent a local minimum or a local maximum. In fact, the function has a point of inflection here. Try to draw a reasonably precise graph for this function in the interval $(-1, 1)$, and observe the shape of the curve.

3.3. Curve Sketching

In this section, we study a few important aspects of curve sketching. In the following, we shall describe a reasonably systematic routine that one may follow. Not every step is applicable to every function. The reader should work through this section and draw the graphs by following the instructions given. We use the convention $y = f(x)$.

STEP 1. SYMMETRY. If the function has some symmetry, then a lot of work can be saved. The two most basic aspects of symmetry are even functions and odd functions.

DEFINITION.
(1) A function f such that $f(-x) = f(x)$ is called an even function.
(2) A function f such that $f(-x) = -f(x)$ is called an odd function.

It is easy to see that the graphs of even functions are symmetric across the vertical axis, whereas the graphs of odd functions are symmetric across the origin.

EXAMPLE 3.3.1. Try to draw the graph of $f(x) = x^2 + 3$. This is an even function.

EXAMPLE 3.3.2. Try to draw the graph of $f(x) = x^3$. This is an odd function.

EXAMPLE 3.3.3. Try to draw the graph of $f(x) = \sin x$. This is an odd function.

EXAMPLE 3.3.4. Try to draw the graph of $f(x) = \cos x$. This is an even function.

STEP 2. PERIODICITY. Certain functions, like $\sin x$ and $\cos x$, have periodicity. We may therefore be able to draw part of the graph, and obtain the rest by repetition.

EXAMPLE 3.3.5. The function $f(x) = \sin x$ has period 2π. Draw the graph in the interval $[0, 2\pi]$. Then complete the graph by repetition.

STEP 3. LOCATING A FEW POINTS OF THE GRAPH. This needs very little explanation!

STEP 4. INTERCEPTS. We may wish to find where the graph of the function intersects the coordinate axes. This may be a simple exercise in some cases, but extremely difficult in other cases.

To see where the graph intersects the y-axis is simple, since the graph intersects the y-axis precisely when $x = 0$. It follows that the graph of the function $f(x)$ intersects the y-axis at the point $(0, f(0))$, provided that the function is defined at $x = 0$.

EXAMPLE 3.3.6. Consider the function $f(x) = x^2 + 2x - 3$. Then $f(0) = -3$, so that the graph of the function intersects the y-axis at the point $(0, -3)$.

To see where the graph intersects the x-axis, we note that this happens precisely when $f(x) = 0$. We therefore need to solve the equation $f(x) = 0$.

EXAMPLE 3.3.7. Consider again the function $f(x) = x^2 + 2x - 3$. To see where the graph intersects the x-axis, we have to solve the equation $x^2 + 2x - 3 = 0$. Now $x^2 + 2x - 3 = (x + 3)(x - 1)$. It follows that the roots are -3 and 1. Hence the graph intersects the x-axis at the points $(-3, 0)$ and $(1, 0)$.

EXAMPLE 3.3.8. If we consider the function $f(x) = x^3 - 3x - 1$, then finding where the graph intersects the x-axis becomes an extremely difficult problem.

Of course, it is not absolutely crucial to locate all the points where the graph of the function intersects the coordinate axes.

STEP 5. ASYMPTOTES. Consider first an example.

EXAMPLE 3.3.9. Try to draw a rough sketch of the graph $y = 1/x$. It is easy to see that the graph gets rather close to the coordinate axes. Try to draw next a rough sketch of the graph $y = 1 + 1/x$. It is easy to see that the graph gets rather close to the lines $x = 0$ and $y = 1$. Such lines are called asymptotes.

We have a horizontal asymptote $y = L$ if

$$\lim_{x\to+\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L.$$

On the other hand, we have a vertical asymptote $x = a$ if

$$\lim_{x\to a} f(x) = +\infty \qquad \text{or} \qquad \lim_{x\to a+} f(x) = +\infty \qquad \text{or} \qquad \lim_{x\to a-} f(x) = +\infty,$$

or if

$$\lim_{x\to a} f(x) = -\infty \qquad \text{or} \qquad \lim_{x\to a+} f(x) = -\infty \qquad \text{or} \qquad \lim_{x\to a-} f(x) = -\infty.$$

While determining the asymptotes, we must also determine which side (possible both) of the asymptote the graph lies.

EXAMPLE 3.3.10. Try to draw the graph of $f(x) = 5 + x^{-3}$. This has horizontal asymptote $y = 5$. Also, $f(x) \to 5$ from above as $x \to +\infty$, and $f(x) \to 5$ from below as $x \to -\infty$. On the other hand, we have

$$\lim_{x\to 0+} f(x) = +\infty \qquad \text{and} \qquad \lim_{x\to 0-} f(x) = -\infty.$$

However, there is insufficient information yet to complete the graph. For example, we have yet to fully understand the behaviour of the function when $x \neq 0$. Does the curve go up and down?

EXAMPLE 3.3.11. Try to draw the graph of $f(x) = 1/(x-1)(x-2)$. This has horizontal asymptote $y = 0$. Also, $f(x) \to 0$ from above as $x \to +\infty$, and $f(x) \to 0$ from above as $x \to -\infty$. On the other hand, we have

$$\lim_{x\to 1-} f(x) = +\infty, \qquad \lim_{x\to 1+} f(x) = -\infty, \qquad \lim_{x\to 2-} f(x) = -\infty \qquad \text{and} \qquad \lim_{x\to 2+} f(x) = +\infty.$$

However, there is insufficient information yet to complete the graph. For example, we have yet to fully understand the behaviour of the function when $1 < x < 2$. Is there any stationary point?

Of course, some graphs may not have any asymptotes at all. However, it is still useful to investigate the behaviour when $x \to -\infty$ and when $x \to +\infty$.

EXAMPLE 3.3.12. Consider again the function $f(x) = x^3 - 3x - 1$. It is easy to check that $f(x) \to +\infty$ as $x \to +\infty$, and that $f(x) \to -\infty$ as $x \to -\infty$. Such information is important.

STEP 6. STATIONARY POINTS. We now use our knowledge on derivatives to further our understanding of the functions. We determine where the function is increasing, where it is decreasing, and locate all the stationary points, local maxima, local minima and points of inflection by using our knowledge of Section 3.2.

EXAMPLE 3.3.13. Let us continue our investigation of the function $f(x) = 5 + x^{-3}$. Simple calculation gives $f'(x) = -3x^{-4}$. It follows that there is no stationary point. Next, note that $f'(x) < 0$ whenever $x \neq 0$. It follows that the function is decreasing in the open intervals $(-\infty, 0)$ and $(0, +\infty)$. We now supplement our earlier effort with this extra information.

EXAMPLE 3.3.14. Let us continue our investigation of the function $f(x) = 1/(x-1)(x-2)$. Simple calculation gives

$$f'(x) = -\frac{2x-3}{(x-1)^2(x-2)^2} \qquad \text{and} \qquad f''(x) = \frac{6x^2-18x+14}{(x-1)^3(x-2)^3}.$$

It follows that there is a stationary point at $x = 3/2$. Furthermore, $f''(3/2) < 0$, so that this stationary point is a local maximum. Note that $f(3/2) = -4$, so that the local maximum is at the point $(3/2, -4)$. Next, note that

$$f'(x) \begin{cases} > 0 & \text{if } x < 3/2 \text{ and } x \neq 1, \\ < 0 & \text{if } x > 3/2 \text{ and } x \neq 2. \end{cases}$$

It follows that the function is increasing in the open intervals $(-\infty, 1)$ and $(1, 3/2)$, and decreasing in the open intervals $(3/2, 2)$ and $(2, +\infty)$. We now supplement our earlier effort with this extra information.

STEP 7. USE OF SECOND DERIVATIVES. If we use the second derivative, we may also be able to see how the tangent to the curve varies. Recall that if $f''(a) > 0$, then $f'(x)$ is increasing at $x = a$, so that the slope of the curve in increasing. On the other hand, if $f''(a) < 0$, then $f'(x)$ is decreasing at $x = a$, so that the slope of the curve is decreasing.

EXAMPLE 3.3.15. Let us return to the function $f(x) = 5 + x^{-3}$ one last time. Simple calculation gives $f''(x) = 12x^{-5}$. It follows that

$$f''(x) \begin{cases} < 0 & \text{if } x < 0, \\ > 0 & \text{if } x > 0. \end{cases}$$

This means that the slope of the tangent decreases when $x < 0$, and increases when $x > 0$. With this extra information, we should get a reasonably good sketch of the graph.

Example 3.3.16. Let us return to the function $f(x) = 1/(x-1)(x-2)$ one last time. Recall that

$$f''(x) = \frac{6x^2 - 18x + 14}{(x-1)^3(x-2)^3} = \frac{6(x-\frac{3}{2})^2 + \frac{1}{2}}{(x-1)^3(x-2)^3},$$

so that

$$f''(x) \begin{cases} > 0 & \text{if } x < 1, \\ < 0 & \text{if } 1 < x < 2, \\ > 0 & \text{if } x > 2. \end{cases}$$

This means that the slope of the tangent increases when $x < 1$ and when $x > 2$, and decreases when $1 < x < 2$. With this extra information, we should get a reasonably good sketch of the graph.

Example 3.3.17. Consider the function $f(x) = x^4 - 2x^3$. Clearly $f(x)$ is not even, odd or periodic, so that Steps 1 and 2 do not apply. For Step 3, we may locate, for example, the points $(-2, 32)$, $(-1, 3)$, $(0, 0)$, $(1, -1)$, $(2, 0)$ and $(3, 27)$. For Step 4, we note that the graph intersects the y-axis at the point $(0, 0)$, and that for the intersection points with the x-axis, we need to solve the equation $x^4 - 2x^3 = 0$, with roots $x = 0$ and $x = 2$. It follows that we have the intersection points $(0, 0)$ and $(2, 0)$. For Step 5, note that $f(x) \to +\infty$ as $x \to +\infty$ or as $x \to -\infty$, so there are no horizontal asymptotes. On the other hand, the function is continuous everywhere, and so there can be no vertical asymptotes. Next, let us consider stationary points. Elementary calculation gives $f'(x) = 4x^3 - 6x^2$ and $f''(x) = 12x^2 - 12x$. It follows that we have stationary points at $x = 0$ and $x = 3/2$. Note that $f''(0) = 0$ and $f''(3/2) > 0$. It follows that the function has a point of inflection at $(0, 0)$ and a local minimum at $(3/2, -27/16)$. On the other hand, note that

$$f'(x) = 2x^2(2x-3) \begin{cases} < 0 & \text{if } x < 3/2, \\ > 0 & \text{if } x > 3/2, \end{cases}$$

so that the function is decreasing in the open interval $(-\infty, 3/2)$ and increasing in the open interval $(3/2, +\infty)$. Finally, note that

$$f''(x) = 12x(x-1) \begin{cases} > 0 & \text{if } x < 0, \\ = 0 & \text{if } x = 0, \\ < 0 & \text{if } 0 < x < 1, \\ = 0 & \text{if } x = 1, \\ > 0 & \text{if } x > 1. \end{cases}$$

This means that the slope of the tangent increases when $x < 0$ or when $x > 1$, and decreases when $0 < x < 1$. It also shows that there is a point of inflection at $(1, -1)$. With all the above information, we should get a reasonably good sketch of the graph.

3.4. Linearization of Error and Approximation of Derivative

To motivate this section, we consider two examples.

Example 3.4.1. Consider again the function $f(x) = x^2$. At any given point x, let us consider a small increment Δx and the behaviour of the function as x changes to $x + \Delta x$. Clearly the value $f(x)$ changes to $f(x + \Delta x)$, giving rise to the error

$$\Delta f = f(x + \Delta x) - f(x) = (x + \Delta x)^2 - x^2 = 2x\Delta x + (\Delta x)^2,$$

and the relative error

$$\frac{\Delta f}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} = 2x + \Delta x.$$

As Δx is taken to be very small, we have respectively the approximations

$$\Delta f \approx 2x\Delta x \qquad \text{and} \qquad \frac{\Delta f}{\Delta x} \approx 2x.$$

Note that the first of these suggests that Δf is essentially directly proportional to Δx, and the second shows that the relative error is an approximation of the derivative.

EXAMPLE 3.4.2. Consider next the function $f(x) = x^3$. At any given point x, let us consider a small increment Δx and the behaviour of the function as x changes to $x + \Delta x$. Clearly the value $f(x)$ changes to $f(x + \Delta x)$, giving rise to the error

$$\Delta f = f(x + \Delta x) - f(x) = (x + \Delta x)^3 - x^3 = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3,$$

and the relative error

$$\frac{\Delta f}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2.$$

As Δx is taken to be very small, we have respectively the approximations

$$\Delta f \approx 3x^2\Delta x \qquad \text{and} \qquad \frac{\Delta f}{\Delta x} \approx 3x^2.$$

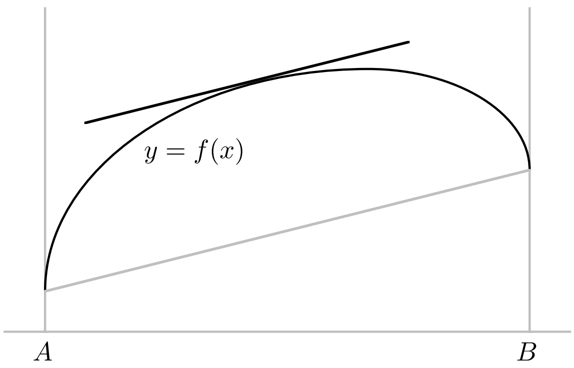
Note again that the first of these suggests that Δf is essentially directly proportional to Δx, and the second shows that the relative error is an approximation of the derivative.

In both of these examples, we clearly have the approximation

$$\Delta f \approx f'(x)\Delta x,$$

demonstrating that Δf is essentially directly proportional to Δx, and with the derivative $f'(x)$ as the proportionality constant. This estimate holds for all functions $f(x)$ that satisfy mild differentiability conditions, demonstrating that the derivative is useful in the study of properties of a function. We shall establish in Chapter 8 the following result which summarizes, with more rigour, this rather precise link.

PROPOSITION 3F. (MEAN VALUE THEOREM) *Suppose that a function $f(x)$ is continuous in the closed interval $[A, B]$, where $A, B \in \mathbb{R}$ with $A < B$. Suppose further that $f'(a)$ exists for every $a \in (A, B)$. Then there exists $c \in (A, B)$ such that $f(B) - f(A) = f'(c)(B - A)$.*



To understand the Mean value theorem, it is easiest to rewrite the conclusion as

$$\frac{f(B) - f(A)}{B - A} = f'(c).$$

The left hand side represents the slope of the line joining the points $(A, f(A))$ and $(B, f(B))$. It follows that the theorem merely says that the tangent to the curve is sometimes parallel to this line.

To illustrate the power of the Mean value theorem, we shall deduce the following simple but powerful consequences.

PROPOSITION 3G. *Suppose that a function $f(x)$ is continuous in the closed interval $[A,B]$, where $A,B \in \mathbb{R}$ with $A<B$. Suppose further that $f'(a)$ exists for every $a \in (A,B)$.*
(a) If $f'(a)=0$ for every $a \in (A,B)$, then $f(x)$ is constant in $[A,B]$.
(b) If $f'(a)>0$ for every $a \in (A,B)$, then $f(x)$ is strictly increasing in $[A,B]$.
(c) If $f'(a)<0$ for every $a \in (A,B)$, then $f(x)$ is strictly decreasing in $[A,B]$.

Proof. Suppose that $A \le x_1 < x_2 \le B$. Applying the Mean value theorem to the function $f(x)$ in the closed interval $[x_1,x_2]$, we have

$$f(x_2)-f(x_1)=(x_2-x_1)f'(c)$$

for some $c \in [x_1,x_2] \subseteq [A,B]$. It follows that

$$f(x_2)-f(x_1) = \begin{cases} =0 & \text{in case (a),} \\ >0 & \text{in case (b),} \\ <0 & \text{in case (c),} \end{cases}$$

giving the desired results. ○

Example 3.4.3. Note that if $f(x)=-x^3$, then $f'(x)=-3x^2<0$ whenever $x \neq 0$. It follows that if $0 \notin (A,B)$, then we can apply Proposition 3G(c) immediately to conclude that $f(x)$ is strictly decreasing in $[A,B]$. On the other hand, if $0 \in (A,B)$, then we can apply Proposition 3G(c) immediately to conclude that $f(x)$ is strictly decreasing in $[A,0]$ and in $[0,B]$. However, if $A \le x_1 < 0 < x_2 \le B$, then we clearly have $f(x_1)>f(0)>f(x_2)$. It follows that $f(x)$ is strictly decreasing in $[A,B]$ for every $A<B$.

Example 3.4.4. Note that if $f(x)=\sin x$, then $f'(x)=\cos x>0$ for every $x \in (-\pi/2,\pi/2)$. It follows from Proposition 3G(b) that $f(x)$ is strictly increasing in the closed interval $[-\pi/2,\pi/2]$.

Example 3.4.5. Consider the function $f(x)=6x+5\cos x$. Then $f'(x)=6-5\sin x>0$ for every $x \in \mathbb{R}$. It follows from Proposition 3G(b) that $f(x)$ is strictly increasing in any closed interval.

3.5. Resolving Indeterminate Limits

Suppose that $f(x)$ and $g(x)$ are differentiable functions with continuous derivatives, and with $f(a)=0$ and $g(a)=0$. To study the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)}, \tag{2}$$

we cannot simply write down the quotient $f(a)/g(a)$, since this is indeterminate. Depending on the nature of the functions $f(x)$ and $g(x)$, there are different techniques which may enable us to determine the limit (2). However, a simple technique is given by the following very useful result, which we shall state without proof.

PROPOSITION 3H. (L'HÔPITAL'S RULE) *Suppose that $f(x) \to f(a)=0$ and $g(x) \to g(a)=0$ as $x \to a$. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}, \tag{3}$$

provided that the limit on the right hand side of (3) exists.

Example 3.5.1. To investigate

$$\lim_{x\to 0}\frac{\sin x}{x},$$

let $f(x)=\sin x$ and $g(x)=x$. Then $f(x)\to 0$ and $g(x)\to 0$ as $x\to 0$. Consider $f'(x)=\cos x$ and $g'(x)=1$ instead, and note that $f'(x)\to 1$ and $g'(x)\to 1$ as $x\to 0$. It follows from the quotient rule that $f'(x)/g'(x)\to 1$ as $x\to 0$. It now follows from Proposition 3H that the origimal limit is equal to 1.

Example 3.5.2. We shall show that

$$\lim_{x\to 0}\frac{2\cos x+x\sin x-2}{x^4}=-\frac{1}{12}.$$

To do this, let $f(x)=2\cos x+x\sin x-2$ and $g(x)=x^4$. Then $f(x)\to 0$ and $g(x)\to 0$ as $x\to 0$. Consider $f'(x)=x\cos x-\sin x$ and $g'(x)=4x^3$ instead. Again, we have $f'(x)\to 0$ and $g'(x)\to 0$ as $x\to 0$. Consider $f''(x)=-x\sin x$ and $g''(x)=12x^2$ instead, and note that

$$\frac{f''(x)}{g''(x)}=-\frac{1}{12}\frac{\sin x}{x}.$$

At this point, we can use the previous example, and conclude that

$$\lim_{x\to 0}\frac{f''(x)}{g''(x)}=-\frac{1}{12}.$$

Applying l'Hôpital's rule once, we have

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)}=\lim_{x\to 0}\frac{f''(x)}{g''(x)}=-\frac{1}{12}. \tag{4}$$

Applying l'Hôpital's rule again and using (4), we have

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{f'(x)}{g'(x)}=-\frac{1}{12}.$$

Remark. Note that l'Hôpital's rule, as stated in Proposition 3H, is useful in resolving indeterminate limits like $0/0$. Similar techniques apply for resolving indeterminate limits like ∞/∞, and also in limiting situations like $x\to a+$, $x\to a-$, $x\to +\infty$ and $x\to\infty$.

Example 3.5.3. We shall show that

$$\lim_{x\to+\infty}\frac{x^2}{(x+1)^2}=1.$$

To do this, let $f(x)=x^2$ and $g(x)=(x+1)^2$. We have $f(x)\to+\infty$ and $g(x)\to+\infty$ as $x\to+\infty$. Consider $f'(x)=2x$ and $g'(x)=2x+2$ instead. Again, we have $f'(x)\to+\infty$ and $g'(x)\to+\infty$ as $x\to+\infty$. Consider $f''(x)=2$ and $g''(x)=2$ instead, and note that

$$\lim_{x\to+\infty}\frac{f''(x)}{g''(x)}=1.$$

Applying l'Hôpital's rule once, we have

$$\lim_{x\to+\infty}\frac{f'(x)}{g'(x)}=\lim_{x\to+\infty}\frac{f''(x)}{g''(x)}=1. \tag{5}$$

Applying l'Hôpital's rule again and using (5), we have

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)}=\lim_{x\to+\infty}\frac{f'(x)}{g'(x)}=1.$$

EXAMPLE 3.5.4. We shall show that

$$\lim_{x\to0+}\frac{x+x^{-1}}{\cot x}=1.$$

To do this, let $f(x)=x+x^{-1}$ and $g(x)=\cot x$. Note that $f(x)\to+\infty$ and $g(x)\to+\infty$ as $x\to0+$. Consider $f'(x)=1-x^{-2}$ and $g'(x)=-\csc^2 x$ instead. Again, we have $f'(x)\to-\infty$ and $g'(x)\to-\infty$ as $x\to0+$. Note, however, that

$$\frac{f'(x)}{g'(x)}=-\frac{1-x^{-2}}{\csc^2 x}=\frac{\sin^2 x}{x^2}-\sin^2 x. \tag{6}$$

Note now that

$$\lim_{x\to0+}\left(\frac{\sin^2 x}{x^2}-\sin^2 x\right)=1. \tag{7}$$

Applying l'Hôpital's rule and using (6) and (7), we have

$$\lim_{x\to0+}\frac{f(x)}{g(x)}=\lim_{x\to0+}\frac{f'(x)}{g'(x)}=1.$$

Try this problem also without l'Hôpital's rule.

3.6. Implicit Differentiation

So far, all our functions have been given by some formula which gives explicitly the value $f(x)$ for every x in the domain. However, there are also instances where such information is given only implicitly. The question is to find the derivative of this function. We shall illustrate the technique by four examples. The advanced reader may choose instead to proceed immediately to the next section.

EXAMPLE 3.6.1. Consider the function $y=f(x)$ described by the curve $9x^2+y^2=25-x$, with $y>0$. Of course, we can write $y=(25-x-9x^2)^{1/2}$, and calculate $\mathrm{d}y/\mathrm{d}x$ accordingly. Alternatively, we can use the Chain rule in the following way. Let $w=y^2$. Then

$$\frac{\mathrm{d}w}{\mathrm{d}x}=\frac{\mathrm{d}w}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x}=2y\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Note, however, that

$$\frac{\mathrm{d}}{\mathrm{d}x}(9x^2+y^2)=\frac{\mathrm{d}}{\mathrm{d}x}(25-x).$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}x}(9x^2+y^2)=\frac{\mathrm{d}}{\mathrm{d}x}(9x^2)+\frac{\mathrm{d}w}{\mathrm{d}x}=18x+2y\frac{\mathrm{d}y}{\mathrm{d}x}\qquad\text{and}\qquad\frac{\mathrm{d}}{\mathrm{d}x}(25-x)=-1,$$

we have

$$18x+2y\frac{\mathrm{d}y}{\mathrm{d}x}=-1.$$

EXAMPLE 3.6.2. Consider the function $y = f(x)$ described by the equation

$$2x^2 y + \cos y = x^3.$$

Here, it is hard, if not impossible, to describe y explicitly in terms of x. However,

$$\frac{\mathrm{d}}{\mathrm{d}x}(2x^2 y + \cos y) = \frac{\mathrm{d}}{\mathrm{d}x}(x^3).$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}x}(2x^2 y + \cos y) = 4xy + 2x^2\frac{\mathrm{d}y}{\mathrm{d}x} - (\sin y)\frac{\mathrm{d}y}{\mathrm{d}x} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(x^3) = 3x^2,$$

we have

$$4xy + (2x^2 - \sin y)\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2.$$

EXAMPLE 3.6.3. We want to find the maximum value and minimum value of

$$z = x + 2y \tag{8}$$

subject to the constraint

$$x^2 + y^2 = 20. \tag{9}$$

Differentiating (8) and (9) with respect to x, we obtain respectively

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 1 + 2\frac{\mathrm{d}y}{\mathrm{d}x}$$

and

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0. \tag{10}$$

When z is maximized or minimized, we must have $\mathrm{d}z/\mathrm{d}x = 0$, so that

$$1 + 2\frac{\mathrm{d}y}{\mathrm{d}x} = 0. \tag{11}$$

Combining (10) and (11) and eliminating $\mathrm{d}y/\mathrm{d}x$, we obtain

$$2x - y = 0. \tag{12}$$

Combining (9) and (12), we obtain $x = \pm 2$. Obviously, $z = 10$ when $x = 2$, while $z = -10$ when $x = -2$. It follows that the maximum value of z is 10, and that the minimum value of z is -10.

Problems for Chapter 3

1. Use the definition of a derivative to show that for the function $f(x) = 3x^{1/3}$, we have $f'(a) = a^{-2/3}$ for every positive $a \in \mathbb{R}$.
[Hint: Use the identity $\alpha^3 - \beta^3 = (\alpha - \beta)(\alpha^2 + \alpha\beta + \beta^2)$.]

2. Suppose that $f(x)$ and $g(x)$ are twice differentiable at $x = a$. Show that

$$(fg)''(a) = f''(a)g(a) + 2f'(a)g'(a) + f(a)g''(a).$$

3. Consider the function $f(x) = |x| - 3$.
   a) Show that $f(x)$ is differentiable at $x = a$ for every non-zero $a \in \mathbb{R}$.
   [Hint: Study the cases $a > 0$ and $a < 0$ separately.]
   b) Comment in view of Proposition 3D.

4. Differentiate each of the following functions:
   a) $f(x) = \sqrt{\sin^{-1} x + 1}$
   b) $f(x) = \dfrac{1}{\tan^{-1}(4x)}$
   c) $f(x) = \dfrac{x}{\sin^{-1} x}$
   d) $f(x) = \mathrm{e}^x \cos^{-1} x$

5. Consider the function $y = \dfrac{x+1}{x^2+4}$.
   a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.
   b) Find the equation of the tangent to the curve at the point $(0, 1/4)$.

6. Determine $a, b, c, d \in \mathbb{R}$ so that the curve $y = ax^3 + bx^2 + cx + d$ passes through the points $(0, 3)$ and $(2, 5)$ and has stationary points at $x = 1/3$ and $x = 1$.

7. Consider the (odd) function $f(x) = x^5 - 10x^3 + 25x$.
   a) Locate the four stationary points by studying $f'(x)$.
   b) Locate the three points of inflection by studying $f''(x)$.
   c) By evaluating the second derivative at the four stationary points, show that $f(x)$ has local maxima at two of these points and local minima at the other two.
   d) Where does the graph of the function intersect the y-axis?
   e) Where does the graph of the function intersect the x-axis?
   f) Study the limits $\displaystyle\lim_{x\to+\infty} f(x)$ and $\displaystyle\lim_{x\to-\infty} f(x)$.
   g) Sketch the graph, clearly marking the intercepts, maxima, minima and points of inflection.

8. Given $y = f(x) = \dfrac{x^2+1}{x^2-1}$.
   a) Find the (largest) domain of $y = f(x)$ as a real valued function, all the stationary points and determine their nature.
   b) Find all the asymptotes of the curve $y = f(x)$.
   c) Sketch the curve and find the range of $y = f(x)$.

9. For each of the following curves, find all the vertical and horizontal asymptotes and sketch the curve:
   a) $y = \dfrac{x}{x^2+1}$
   b) $y = \dfrac{x+1}{x^2-9}$
   c) $y = \dfrac{x^2+1}{(x-2)(x-4)}$

10. Sketch each of the following curves, clearly marking the maxima and minima:
   a) $y = \dfrac{x}{x^2+1}$
   b) $y = \dfrac{x+1}{x^2-9}$
   c) $y = \dfrac{x^2-3}{x-2}$

11. Let $f(x) = \sqrt{x}$ and $g(x) = \dfrac{x}{x^2 - 1}$.
    a) Find the largest possible domains of $f(x)$ and $g(x)$ as real valued functions.
    b) Find the composite functions $(f \circ g)(x) = f(g(x))$ and $(g \circ f)(x) = g(f(x))$.
    c) Find the inverse function of $f(x)$.
    d) Find all the asymptotes of $g(x)$.
    e) Find the derivative of $g(x)$. Is $g(x)$ a monotonic function in the interval $(-1, 1)$; in other words, is $g(x)$ always increasing or always decreasing in the interval $(-1, 1)$? Give your reasons.
    f) Sketch the curve $y = g(x)$.

12. Consider the function $f(x) = \dfrac{x^2 + 3x - 3}{x - 1}$, continuous everywhere except at $x = 1$.
    a) Show that

$$f'(x) \begin{cases} > 0 & \text{if } x < 0, \\ = 0 & \text{if } x = 0, \\ < 0 & \text{if } 0 < x < 1, \\ < 0 & \text{if } 1 < x < 2, \\ = 0 & \text{if } x = 2, \\ > 0 & \text{if } x > 2. \end{cases}$$

    b) Show that

$$f''(x) \begin{cases} < 0 & \text{if } x < 1, \\ > 0 & \text{if } x > 1. \end{cases}$$

    c) Explain why $f(x)$ has a local maximum at $x = 0$ and a local minimum at $x = 2$.
    d) Explain why $f(x)$ has no points of inflection.
    e) Use the Mean value theorem to explain why $f(x) < f(0)$ when $0 < x < 1$.
    f) We have $f(0) > 0$ and $f(-1) < 0$. Since the function is continuous in the interval $[-1, 0]$, so that we can draw its graph on this interval without lifting the pen from paper, there exists $c \in (-1, 0)$ such that $f(c) = 0$. Use the Mean value theorem to explain why there is no other real number $c < 0$ such that $f(c) = 0$.
    g) Sketch the graph of $f(x)$, clearly indicating the local maximum, the local minimum, the real number c in part (f). You may use the following additional information:

$$\begin{aligned} f(x) &\to -\infty \quad \text{as } x \to 1- \text{ or } x \to -\infty, \\ f(x) &\to +\infty \quad \text{as } x \to 1+ \text{ or } x \to +\infty. \end{aligned}$$

13. The function $y = f(x)$ is given implicitly by the equation $2x^2 + y^3 = 9$.
    a) Find the first derivative $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at the point $(-2, 1)$.
    b) Find the coordinates of the point(s) on the curve where the tangent(s) is (are) horizontal.
    c) Find the second derivative $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at the point $(-2, 1)$.

14. For each of the following, find the first derivative y' and second derivative y'' in terms of x and y by implicit differentiation:
    a) $x^3 + y^3 = 1$ b) $\sqrt{x} + \sqrt{y} = 1$ c) $x^2 + 6xy + y^2 = 8$

15. The function $y = f(x)$ is defined implicitly by $\sin x + y^3 = 8$. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at the point $(0, 2)$.

16. The function $y = f(x)$ is given implicitly by $x^3y + y^3 = 9$. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at the point $(2, 1)$.

17. Consider the ellipse $(x+1)^2 + 2(y-1)^2 = 6$.
   a) Determine the slope of the tangent at the point $(1,2)$.
   b) Determine the slope of the tangent at the point $(1,0)$.

18. Use L'Hopital's rule to find each of the following:
   a) $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x^3}$
   b) $\displaystyle\lim_{x\to 0} \frac{\tan x - x}{x^3}$
   c) $\displaystyle\lim_{x\to 1} \frac{x^3 - 3x + 2}{x^3 + x^2 - 5x + 3}$