FIRST YEAR CALCULUS

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Chapter 4

SOME SPECIAL FUNCTIONS

4.1. Exponential Functions

In this section, we construct a class of functions of the form $f_a : \mathbb{R} \to \mathbb{R}$, where for every $x \in \mathbb{R}$,

$$f_a(x) = a^x.$$

Here $a > 0$ denotes a positive real constant.

Let us state very carefully what we mean by a^x. We would like to define a^x appropriately so that

$$a^{x+y} = a^x a^y$$

for every $x, y \in \mathbb{R}$. To do so, we must have $a^{x+0} = a^x a^0$. This forces us to write

$$a^0 = 1. \tag{1}$$

Also, it seems reasonable to write

$$a^n = \underbrace{a \ldots a}_{n \text{ times}} \quad \text{for every } n \in \mathbb{N}. \tag{2}$$

Next, it is clear that it is necessary to define, for every $p, q \in \mathbb{N}$,

$$y = a^{1/q} > 0 \qquad \text{if and only if} \qquad y^q = a, \tag{3}$$

and

$$a^{p/q} = (a^{1/q})^p. \tag{4}$$

Note that (2)–(4) give a^x for every $x \in \mathbb{Q}^+$, the set of all positive rational numbers. Our definition is now extended to cover the set all negative rational numbers $\mathbb{Q}^-$ by

$$a^x = \frac{1}{a^{-x}} \quad \text{for every } x \in \mathbb{Q}^-. \tag{5}$$

Hence we have, by (1)–(5), defined a^x for every $x \in \mathbb{Q}$.

The question that remains is how we define a^x when x is irrational. Without giving all the details, we claim that it is possible to define a^x for all irrational numbers x so that the function $f_a(x) = a^x$ is continuous and differentiable everywhere in $\mathbb{R}$. In other words, we can draw the graph without lifting pen from paper, and the tangent exists everywhere.

Now let us consider the derivative $f_a'(x)$. Clearly

$$f_a'(x) = \lim_{y \to x} \frac{a^y - a^x}{y - x} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = f_a(x) \lim_{h \to 0} \frac{a^h - 1}{h}.$$

Let us write

$$c(a) = \lim_{h \to 0} \frac{a^h - 1}{h}.$$

Numerical evidence suggests that $c(2) < 1$ and $c(3) > 1$. Indeed, it can be shown that there exists a unique $\mathrm{e} \in (2, 3)$ such that $c(\mathrm{e}) = 1$.

With this number e, we have the function $f : \mathbb{R} \to \mathbb{R}$, where for every $x \in \mathbb{R}$,

$$f(x) = \mathrm{e}^x.$$

The results below are easy consequences of our discussion.

PROPOSITION 4A. *The function $f : \mathbb{R} \to \mathbb{R}$, defined for every $x \in \mathbb{R}$ by $f(x) = \mathrm{e}^x$, has the following properties:*
(a) $f(x) > 0$ *for every* $x \in \mathbb{R}$*, and* $f(0) = 1$.
(b) $f(x_1 + x_2) = f(x_1)f(x_2)$ *for every* $x_1, x_2 \in \mathbb{R}$.
(c) $f(x)$ *is differentiable, and* $f'(x) = f(x)$ *for every* $x \in \mathbb{R}$.
(d) $f(x)$ *is strictly increasing in* $\mathbb{R}$*; in other words,* $f(x_1) < f(x_2)$ *whenever* $x_1 < x_2$.
(e) $f(x) \to 0$ *as* $x \to -\infty$.
(f) $f(x) \to +\infty$ *as* $x \to +\infty$.

4.2. The Exponential and Logarithmic Functions

It is easy to see that the function considered in Proposition 4A is one-to-one, in view of part (d). On the other hand, the function is not onto, in view of part (a). However, this "mishap" can be corrected easily by changing the codomain to $\mathbb{R}^+ = f(\mathbb{R})$, the set of all positive real numbers. So let us change the codomain.

PROPOSITION 4B. *The function* $\exp : \mathbb{R} \to \mathbb{R}^+$*, defined for every* $x \in \mathbb{R}$ *by* $\exp(x) = \mathrm{e}^x$*, is one-to-one and onto.*

DEFINITION. The function $\exp(x)$ is usually called the exponential function.

It now follows from Proposition 2C that the exponential function $\exp : \mathbb{R} \to \mathbb{R}^+$ has an inverse function. This is known as the logarithmic function, and denoted by $\log : \mathbb{R}^+ \to \mathbb{R}$. Hence

$$y = \exp(x) \qquad \text{if and only if} \qquad x = \log(y).$$

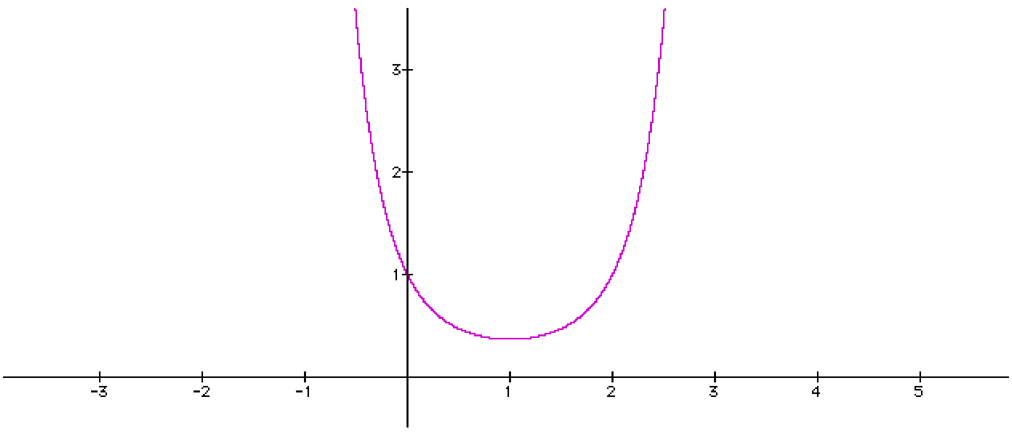
The results below are easy consequences of our discussion.

PROPOSITION 4C. *The logarithmic function* $\log : \mathbb{R}^+ \to \mathbb{R}$ *has the following properties:*
(a) $\log(y) > 0$ *for every* $y > 1$*,* $\log(y) < 0$ *for every positive* $y < 1$*, and* $\log(1) = 0$*.*
(b) $\log(y_1 y_2) = \log(y_1) + \log(y_2)$ *for every* $y_1, y_2 \in \mathbb{R}^+$*.*
(c) $\log(y)$ *is differentiable, and* $\log'(y) = 1/y$ *for every* $y \in \mathbb{R}^+$*.*
(d) $\log(y)$ *is strictly increasing in* $\mathbb{R}^+$*; in other words,* $\log(y_1) < \log(y_2)$ *whenever* $0 < y_1 < y_2$*.*
(e) $\log(y) \to -\infty$ *as* $y \to 0+$*.*
(f) $\log(y) \to +\infty$ *as* $y \to +\infty$*.*

The only difficult part is (c). Here we can use the result $\mathrm{d}x/\mathrm{d}y = 1/(\mathrm{d}y/\mathrm{d}x)$. Then if $x = \log(y)$, then $y = \exp(x)$; hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \exp(x) = y \qquad \text{and} \qquad \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{y}.$$

Example 4.2.1. Consider the function $f(x) = \mathrm{e}^{x^2-2x}$.

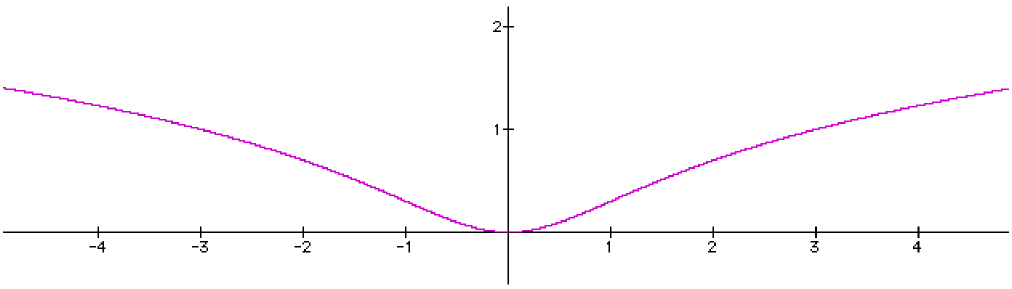


The graph does not intersect the x-axis, and intersects the y-axis at the point $(0,1)$. Also, $f(x) \to +\infty$ as $x \to +\infty$ and as $x \to -\infty$. On the other hand, it follows from the Chain rule that

$$f'(x) = (2x-2)\mathrm{e}^{x^2-2x} \begin{cases} < 0 & \text{if } x < 1, \\ = 0 & \text{if } x = 1, \\ > 0 & \text{if } x > 1. \end{cases}$$

Hence there is a stationary point at $x = 1$. Also the function is decreasing when $x < 1$ and increasing when $x > 1$. Now $f''(x) = ((2x-2)^2 + 2)\mathrm{e}^{x^2-2x} > 0$ always. It follows that the function has a local minimum at $x = 1$. Furthermore, the slope of the tangent is always increasing.

Example 4.2.2. Consider the (even) function $f(x) = \log(x^2 + 1)$.



The graph intersects the coordinate axes at the point $(0,0)$. Also, $f(x) \to +\infty$ as $x \to +\infty$ and as $x \to -\infty$. On the other hand, it follows from the Chain rule that

$$f'(x) = \frac{2x}{x^2+1} \begin{cases} < 0 & \text{if } x < 0, \\ = 0 & \text{if } x = 0, \\ > 0 & \text{if } x > 0. \end{cases}$$

Hence there is a stationary point at $x = 0$. Also the function is decreasing when $x < 0$ and increasing when $x > 0$. Now

$$f''(x) = \frac{2-2x^2}{(x^2+1)^2} \begin{cases} < 0 & \text{if } x < -1, \\ = 0 & \text{if } x = -1, \\ > 0 & \text{if } -1 < x < 1, \\ = 0 & \text{if } x = 1, \\ < 0 & \text{if } x > 1. \end{cases}$$

It follows that the function has a local minimum at $x = 0$. Also it has points of inflection at $x = -1$ and at $x = 1$. Furthermore, the slope of the curve is decreasing in the intervals $(-\infty, -1)$ and $(1, \infty)$, and increasing in the interval $(-1, 1)$.

4.3. Derivatives of the Inverse Trigonometric Functions

The purpose of this section is to determine the derivatives of the inverse trigonometric functions by using implicit differentiation and our knowledge on the derivatives of the trigonometric functions.

For notational purposes, we shall write

$$y = \sin^{-1} x \qquad \text{if and only if} \qquad x = \sin y,$$

and similarly for the other trigonometric functions. The six inverse trigonometric functions are well defined, provided that we restrict the values for x to suitable intervals of real numbers. For simplicity, we shall assume that $0 < y < \pi/2$, so that y is in the first quadrant, and so all the trigonometric functions have positive values.

Example 4.3.1. If $y = \sin^{-1} x$, then $x = \sin y$. Differentiating with respect to x, we obtain

$$1 = \cos y \frac{\mathrm{d}y}{\mathrm{d}x}, \qquad \text{so that} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

If $y = \cos^{-1} x$, then $x = \cos y$. Differentiating with respect to x, we obtain

$$1 = -\sin y \frac{\mathrm{d}y}{\mathrm{d}x}, \qquad \text{so that} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}}.$$

Example 4.3.2. If $y = \tan^{-1} x$, then $x = \tan y$. Differentiating with respect to x, we obtain

$$1 = \sec^2 y \frac{\mathrm{d}y}{\mathrm{d}x}, \qquad \text{so that} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

If $y = \cot^{-1} x$, then $x = \cot y$. Differentiating with respect to x, we obtain

$$1 = -\csc^2 y \frac{\mathrm{d}y}{\mathrm{d}x}, \qquad \text{so that} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\csc^2 y} = -\frac{1}{1+\cot^2 y} = -\frac{1}{1+x^2}.$$

EXAMPLE 4.3.3. If $y = \sec^{-1} x$, then $x = \sec y$. Differentiating with respect to x, we obtain

$$1 = \tan y \sec y \, \frac{\mathrm{d}y}{\mathrm{d}x}, \qquad \text{so that} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\tan y \sec y} = \frac{1}{(\sec^2 y - 1)^{1/2} \sec y} = \frac{1}{x\sqrt{x^2 - 1}}.$$

If $y = \csc^{-1} x$, then $x = \csc y$. Differentiating with respect to x, we obtain

$$1 = -\cot y \csc y \, \frac{\mathrm{d}y}{\mathrm{d}x}, \qquad \text{so that} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\cot y \csc y} = -\frac{1}{(\csc^2 y - 1)^{1/2} \csc y} = -\frac{1}{x\sqrt{x^2 - 1}}.$$

4.4. Rates of Growth of some Special Functions

In this last section, we study a few classes of special functions where the choice of the parameters play a key role in their rate of growth. These functions are all exponential in nature.

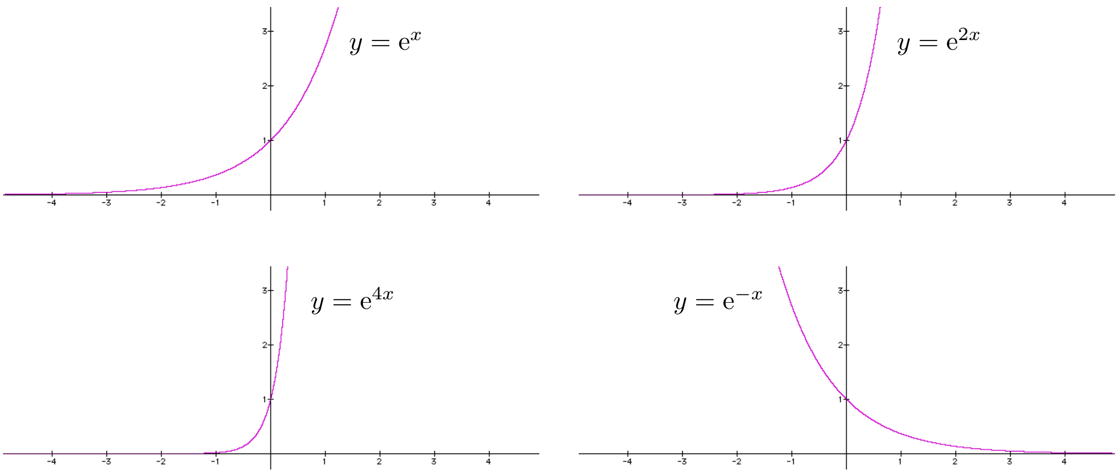
EXAMPLE 4.4.1. Consider the function $f(x) = \mathrm{e}^{kx}$, where $k \in \mathbb{R}$ is fixed. Clearly

$$f'(x) = k\mathrm{e}^{kx} = kf(x),$$

so the growth is proportional to its size. Note also that

$$f'(x) \begin{cases} > 0 & \text{if } k > 0, \\ < 0 & \text{if } k < 0. \end{cases}$$

Below we show the graphs in the cases $k = 1$, $k = 2$, $k = 4$ and $k = -1$.



Note that the graphs for $k = 1$ and $k = -1$ are images of each other across the vertical axis. What happens in the case $k = 0$?

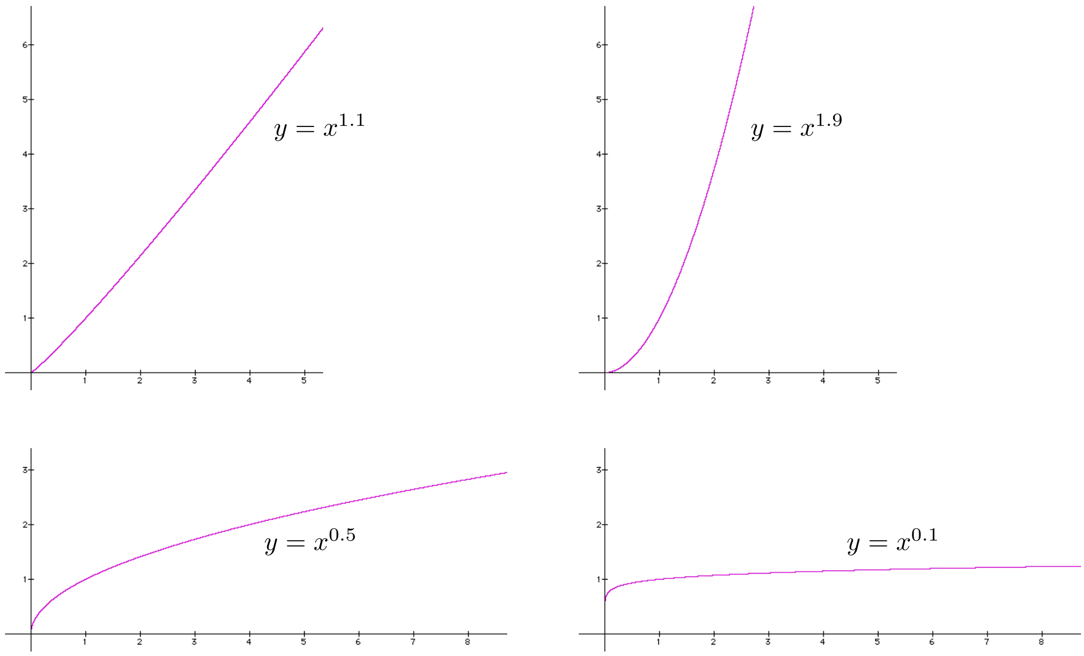
EXAMPLE 4.4.2. Consider the function $f(x) = x^\alpha$, where $\alpha \in \mathbb{R}$ is fixed. Then $f(x) = \mathrm{e}^{\alpha \log x}$, and so

$$f'(x) = \frac{\alpha}{x} \mathrm{e}^{\alpha \log x} = \frac{\alpha}{x} f(x).$$

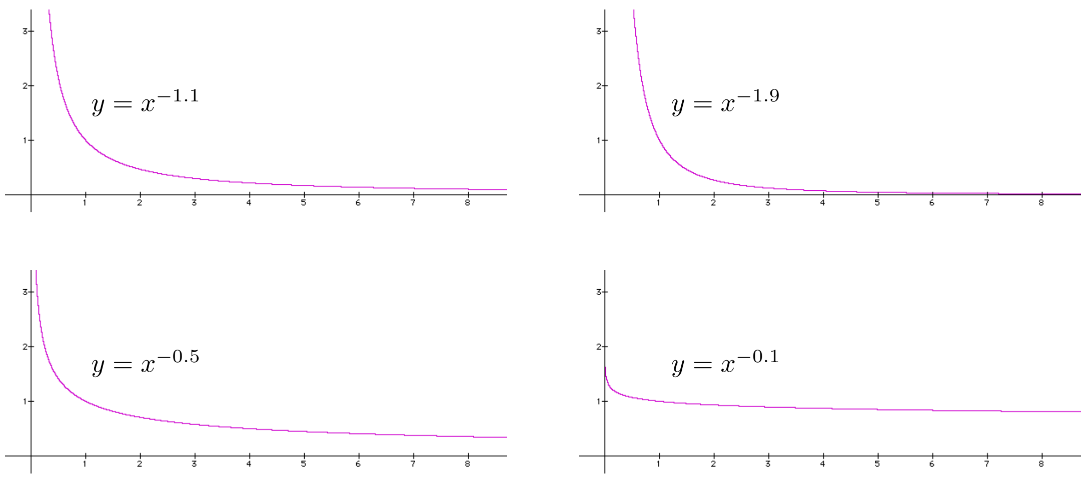
Here we are only interested in the range $x > 0$, so that

$$f'(x) \begin{cases} > 0 & \text{if } \alpha > 0, \\ < 0 & \text{if } \alpha < 0. \end{cases}$$

Below we show the graphs in the cases $\alpha = 1.1$, $\alpha = 1.9$, $\alpha = 0.5$ and $\alpha = 0.1$ where the functions are increasing.



Below we show the graphs in the cases $\alpha = -1.1$, $\alpha = -1.9$, $\alpha = -0.5$ and $\alpha = -0.1$ where the functions are decreasing.



What happens in the case $\alpha = 0$?

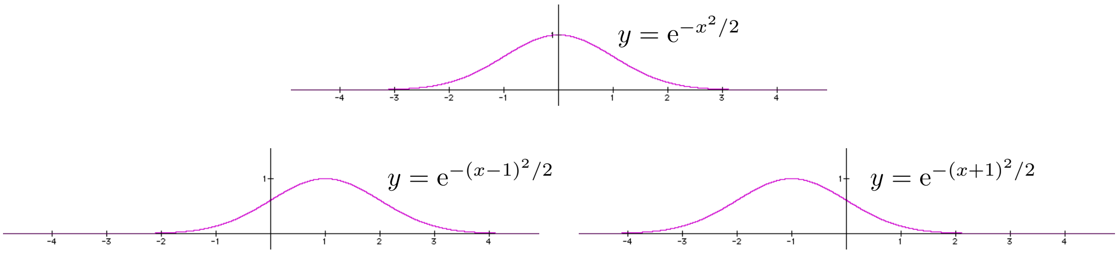
EXAMPLE 4.4.3. Consider the function $f(x) = \mathrm{e}^{-(x-a)^2/b}$, where $a, b \in \mathbb{R}$ are fixed and $b > 0$. Then

$$f'(x) = -\frac{2(x-a)}{b}\,\mathrm{e}^{-(x-a)^2/b} = -\frac{2(x-a)}{b}\,f(x).$$

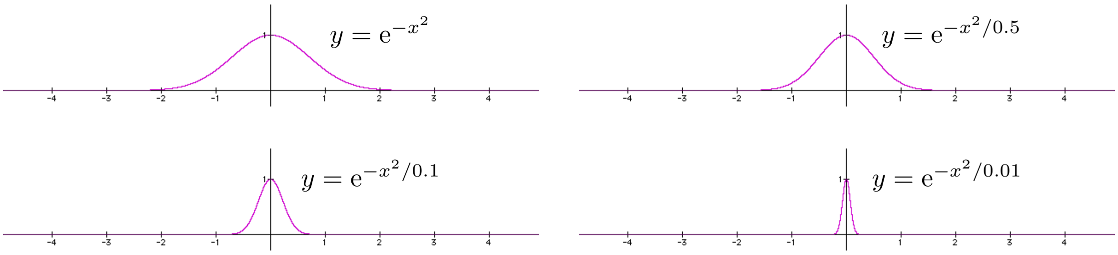
It is easy to see that

$$f'(x) \begin{cases} > 0 & \text{if } x < a, \\ < 0 & \text{if } x > a. \end{cases}$$

Below we show the graphs in the cases $a = 0$, $a = 1$ and $a = -1$, with the same value of $b = 2$.



Note that the shape of the graph is independent of the choice of the paramater a which in fact determines the horizontal positioning of the graph. On the other hand, the rate of growth of the function is determined by the value of the parameter b. Below we show the graphs in the cases $b = 1$, $b = 0.5$, $b = 0.1$ and $b = 0.01$, with the same value of $a = 0$.



What happens if the parameter b is a very large positive number?

Example 4.4.4. Consider the function $f(x) = a^{b^x}$, where $a, b \in \mathbb{R}$ are positive and fixed. Then

$$f(x) = e^{b^x \log a} = e^{e^{x \log b} \log a},$$

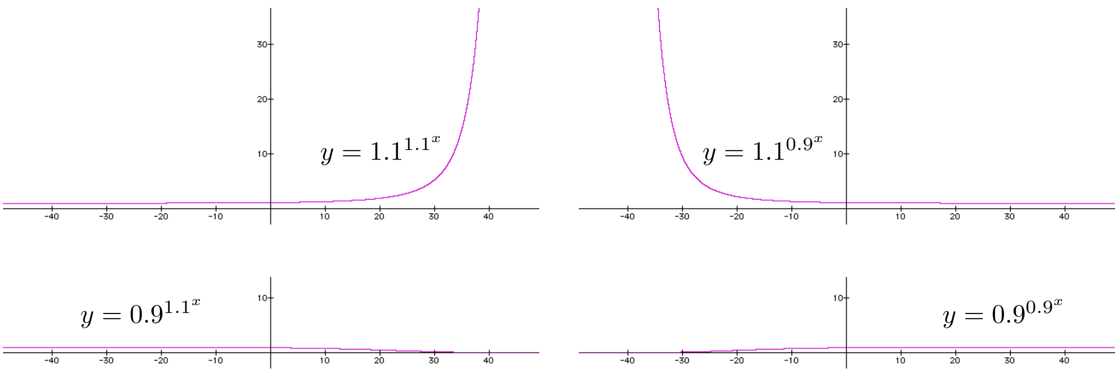
so that

$$f'(x) = (\log a)(\log b)e^{x \log b}e^{e^{x \log b} \log a} = (\log a)(\log b)b^x a^{b^x} = (\log a)(\log b)b^x f(x).$$

Note also that

$$f'(x) \begin{cases} > 0 & \text{if } (\log a)(\log b) > 0, \\ < 0 & \text{if } (\log a)(\log b) < 0. \end{cases}$$

The behaviour of the function is very sensitive to small changes of a and b from the value 1. Below we show the graphs in the cases $b = 1.1$ and $b = 0.9$, with $a = 1.1$ and $a = 0.9$.



What happens in the case $a = b = 1$?

Problems for Chapter 4

1. Find a largest domain, the corresponding range and derivative of each of the following functions:
   a) $f(x) = 4\sin^{-1}(5x)$ b) $f(x) = \frac{1}{2}\cos^{-1}(x^2)$ c) $f(x) = \tan^{-1}(x^2 - 1)$

2. Let $f(x) = \dfrac{x+1}{x-1}$. Find the inverse function of $f(x)$ if it exists.

3. Does the function $y = x^3 - 2$ have an inverse function on the real line? Give your reasons. If yes, then find also the inverse function.

4. For each of the following functions, find the inverse if it exists:
   a) $f(x) = x^3 + 1$ b) $f(x) = \dfrac{x+2}{2x-1}$
   c) $f(x) = x^2 - 4x + 3$ on domain $\{x \in \mathbb{R} : x \geq 2\}$