FIRST YEAR CALCULUS

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Chapter 5

APPLICATIONS OF DERIVATIVES

5.1. Kinematics on a Line

In this section, we discuss briefly the ideas of displacement, velocity and acceleration of a particle moving along a line.

Usually, we take time t as the independent variable, and write $x = x(t)$ to denote the displacement of the particle at time t from a fixed position, usually known as the origin. We adopt the usual convention that for horizontal displacement, $x > 0$ denotes a displacement to the right and $x < 0$ denotes a displacement to the left; whereas for vertical displacement, $x > 0$ denotes a displacement upwards and $x < 0$ denotes a displacement downwards.

Suppose that there is a small change Δt in time, resulting in a corresponding change in displacement of $\Delta x = x(t + \Delta t) - x(t)$. Then the average rate of change in displacement over this short period is given by

$$\frac{\Delta x}{\Delta t} = \frac{x(t + \Delta t) - x(t)}{\Delta t}.$$

Letting $\Delta t \to 0$, we obtain the instantaneous rate of change in displacement at time t, called the veocity at time t and denoted by $v = v(t)$, where

$$v(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}.$$

We also use the notation $\dot{x} = x'(t)$ to denote the same quantity. We adopt the usual convention that for horizontal velocity, $v > 0$ denotes moving to the right and $v < 0$ denotes moving to the left; whereas for vertical velocity, $v > 0$ denotes moving upwards and $v < 0$ denotes moving downwards. Naturally, $v = 0$ indicates that the particle is instantaneously stationary.

EXAMPLE 5.1.1. Suppose that $x = x(t) = t^3 + t^2 - 5t + 7$. Then $v = v(t) = x'(t) = 3t^2 + 2t - 5$. We shall assume that the motion is horizontal. At time $t = 0$, we have $x = 7$ and $v = -5$, so that the particle is moving to the left initially. At time $t = 1$, we have $x = 4$ and $v = 0$, so that the particle is stationary at that moment. At time $t = 2$, we have $x = 9$ and $v = 11$, so that the particle is moving to the right.

Suppose that there is a small change Δt in time, resulting in a corresponding change in velocity of $\Delta v = v(t + \Delta t) - v(t)$. Then the average rate of change in velocity over this short period is given by

$$\frac{\Delta v}{\Delta t} = \frac{v(t + \Delta t) - v(t)}{\Delta t}.$$

Letting $\Delta t \to 0$, we obtain the instantaneous rate of change in velocity at time t, called the acceleration at time t and denoted by $a = a(t)$, where

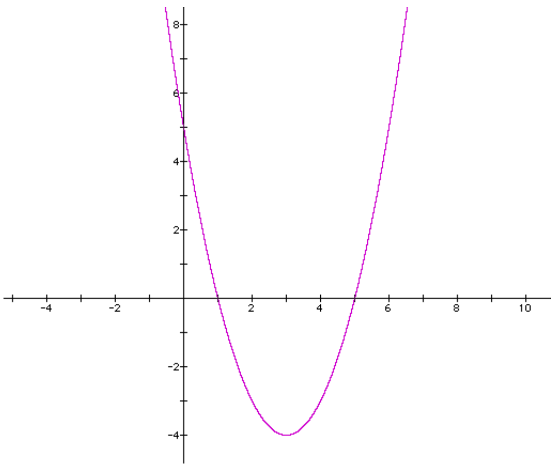
$$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}.$$

We also use the notation $\ddot{x} = x''(t)$ to denote the same quantity. We adopt the usual convention that for horizontal acceleration, $a > 0$ denotes acceleration to the right and $a < 0$ denotes acceleration to the left; whereas for vertical acceleration, $a > 0$ denotes acceleration upwards and $a < 0$ denotes acceleration downwards. Naturally, $a = 0$ indicates that the particle is moving instantaneously at constant speed. Note also that

$$a(t) = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}.$$

EXAMPLE 5.1.2. Let us continue with Example 5.1.1. Clearly $a = a(t) = x''(t) = 6t + 2$, so that we have $a = 2$, $a = 8$ and $a = 14$ at times $t = 0$, $t = 1$ and $t = 2$ respectively. The particle is accelerating towards the right.

EXAMPLE 5.1.3. Suppose that $x(t) = t^2 - 6t + 5$, moving along a horizontal line. We have the graph below.



We have $v(t) = x'(t) = 2t - 6$, so that $v(t) = 0$ when $t = 3$, indicating that the particle is instantaneously stationary at $t = 3$ at displacement $x(3) = -4$. We also note that $x = 0$ when $t = 1$ and $t = 5$, indicating that the particle is at the origin at these times. We have $x > 0$ when $t < 1$ or $t > 5$, indicating that the particle is to the right of the origin; and $x < 0$ when $1 < t < 5$, indicating that the particle is to the left of the origin. We also have $v > 0$ when $t > 3$, indicating acceleration to the right; and $v < 0$ when $t < 3$, indicating acceleration to the left. Finally, note that $a = 2$, so that acceleration is constant and to the right.

REMARK. We make the observation that using the Chain rule, we have

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}x} = v\frac{\mathrm{d}v}{\mathrm{d}x}.$$

This gives acceleration in terms of velocity and the rate of change of velocity with respect to displacement. This observation is useful in some instances.

We conclude this section with a brief discussion of simple harmonic motion, the type of motion that governs the movement of pendulums and waves, amongst other things. A motion governed by a displacement function of the form

$$x(t) = a\cos b(t - t_0) \qquad \text{or} \qquad x(t) = a\sin b(t - t_0),$$

where $a, b \in \mathbb{R}$ are fixed and non-zero, is called simple harmonic motion, with amplitude $|a|$ and period $2\pi/|b|$. The constant t_0 is usually called the phase constant.

Note that in the case when $x(t) = a\cos b(t-t_0)$, we have $\dot{x} = -ab\sin b(t-t_0)$ and $\ddot{x} = -ab^2\cos b(t-t_0)$, so that the motion satisfies the second order ordinary differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - b^2x = 0. \tag{1}$$

It is easily checked that this equation is also satisfied by $x(t) = a\sin b(t - t_0)$. Strictly speaking, every solution $x(t)$ of the equation (1) represents simple harmonic motion.

EXAMPLE 5.1.4. Let $x(t) = a_1\cos bt + a_2\sin bt$, where $a_1, a_2 \in \mathbb{R}$ are fixed and not both zero and where $b \in \mathbb{R}$ is fixed and non-zero. We can choose $T_0 \in \mathbb{R}$ to satisfy

$$\frac{a_1}{\sqrt{a_1^2 + a_2^2}} = \cos T_0 \qquad \text{and} \qquad \frac{a_2}{\sqrt{a_1^2 + a_2^2}} = \sin T_0.$$

Then

$$x(t) = \sqrt{a_1^2 + a_2^2}(\cos bt\cos T_0 + \sin bt\sin T_0) = \sqrt{a_1^2 + a_2^2}\cos(bt - T_0) = \sqrt{a_1^2 + a_2^2}\cos b\left(t - \frac{T_0}{b}\right),$$

representing simple harmonic motion with amplitude $\sqrt{a_1^2 + a_2^2}$ and period $2\pi/|b|$. The quantity T_0/b is the phase constant.

EXAMPLE 5.1.5. Let $x(t) = x_0 + a\cos b(t - t_0)$. This does not strictly describe simple harmonic motion as described above. However, if we make a substitution $X(t) = x(t) - x_0$, then the motion can now be described by $X(t) = a\cos b(t - t_0)$.

5.2. Cost and Revenue Analysis

In this section, we use derivatives to study some problems arising from economics, in connection with cost and revenue analysis.

Let $C(x)$ denote the cost function, representing the cost of producing x units of a particular product. Suppose that production increases by Δx, resulting in an increase $\Delta C = C(x + \Delta x) - C(x)$ in cost. Then the ratio

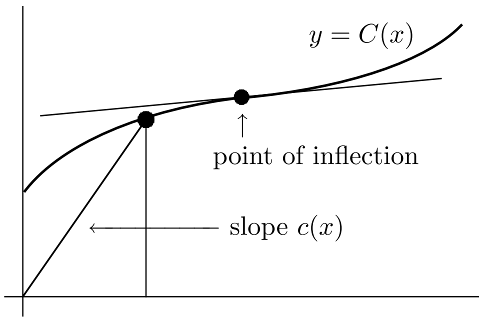
$$\frac{\Delta C}{\Delta x} = \frac{C(x + \Delta x) - C(x)}{\Delta x}$$

represents the average rate of change of cost. If Δx is very small (noting that we are in an idealized situation), then the marginal cost function

$$C'(x) = \frac{\mathrm{d}C}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x}$$

represents the instantaneous rate of change of cost.

In many instances, the cost can be modelled by a cost function of the form $C(x) = a + bx + cx^2 + dx^3$. Here the term a represents the overhead cost like rent and maintenance, the term bx may represent the cost of raw material, and the terms $cx^2 + dx^3$ may represent extra labour cost such as overtime. Usually the marginal cost function $C'(x)$ is decreasing for small values of x due to the economy of scale, and is increasing for large values of x due to overtime cost and inefficiency of a large scale operation. This change gives rise to a point of inflection on the graph for the function $C(x)$.



We also consider the average cost function

$$c(x) = \frac{C(x)}{x},$$

representing the cost per unit if x units are produced. It is easy to see that this is the slope of the line segment joining the point $(x, C(x))$ to the origin, as shown in the diagram above. Note that

$$c'(x) = \frac{xC'(x) - C(x)}{x^2} = 0$$

if $xC'(x) = C(x)$; in other words, if $C'(x) = c(x)$. We therefore conclude that

if the average cost is minimum, then the marginal cost is equal to the average cost.

We also consider the revenue function $R(x)$, representing from the income from the sale of x units of the product. The derivative $R'(x)$ is called the marginal revenue function. The price function

$$p(x) = \frac{R(x)}{x}$$

represents the average income from the sale of each unit, assuming that x units are sold. Usually the price function is a decreasing function.

The function $P(x) = R(x) - C(x)$ is naturally called the profit function, and the derivative $P'(x)$ is called the marginal profit function. Note that $P'(x) = R'(x) - C'(x) = 0$ if $R'(x) = C'(x)$. We therefore conclude that

if the profit is maximum, then the marginal revenue is equal to the marginal cost.

EXAMPLE 5.2.1. A publisher of a calculus textbook works with a cost function

$$C(x) = 50000 + 20x - \tfrac{1}{10000}x^2 + \tfrac{1}{300000000}x^3$$

and a price function $p(x) = 120 - \frac{1}{10000}x$, both in dollars. Clearly we have

$$C'(x) = 20 - \tfrac{1}{5000}x + \tfrac{1}{100000000}x^2 \qquad \text{and} \qquad C''(x) = -\tfrac{1}{5000} + \tfrac{1}{50000000}x,$$

so that $C''(x) = 0$ when $x = 10000$. This means that the marginal cost increases after 10000 copies. On the other hand, we have

$$R(x) = 120x - \tfrac{1}{10000}x^2 \qquad \text{and} \qquad R'(x) = 120 - \tfrac{1}{5000}x.$$

Maximum profit occurs when $R'(x) = C'(x)$, so that

$$120 - \tfrac{1}{5000}x = 20 - \tfrac{1}{5000}x + \tfrac{1}{100000000}x^2,$$

with solution $x = 100000$. This means that maximum profit occurs when exactly 100000 copies are produced and sold. The income is then $R(100000) = 11000000$ dollars at $p(100000) = 110$ dollars per copy. The cost is $C(100000) = 4383333\frac{1}{3}$ dollars, and the profit is $P(100000) = 6616666\frac{2}{3}$ dollars. In other words, the profit is over 66 dollars per copy.

5.3. Modelling with Maxima and Minima

Many practical problems can be understood by suitable mathematical modelling and solution of the underlying mathematical problems. Here we are concerned with a type of such problems where the underlying mathematical problem involves a study of local maxima and minima.

Mathematical modelling is not an exact science, so perhaps we try to adopt a strategy such as this: (1) Identify what we want to maximize or minimize. (2) Express the quantity we wish to maximize or minimize as a function of one other quantity x. (3) Determine all points $x \in \mathbb{R}$ such that $f'(x) = 0$. (4) Decide whether the maximum or minimum in question occurs at one of the solutions in the previous step. Sometimes, it is clear from the nature of the problem that a maximum or minimum exists, so the situation will be clear if there is only one solution arising from the previous step. If there are more solutions in the previous step, then some extra care needs to be exercised.

EXAMPLE 5.3.1. We wish to find two real numbers with sum 10 and with product as large as possible. In this case, we write x as one of the numbers, so that the other number must be $10 - x$, since their sum is 10. We now wish to maximize the product $f(x) = x(10 - x)$. It is easy to see that $f'(x) = 10 - 2x = 0$ when $x = 5$, so the two numbers are $x = 5$ and $10 - x = 5$, with product 25.

EXAMPLE 5.3.2. Metal cans are to be made with fixed volume V. We wish to find the ratio between the height h and the radius r of the base so as to minimize the amount of metal used, with the understanding that the side and ends are made of the same metal. In this case, we wish to minimize the surface area $S = 2\pi r^2 + 2\pi rh$, but the two parameters r and h are related by $V = \pi r^2 h$, so that

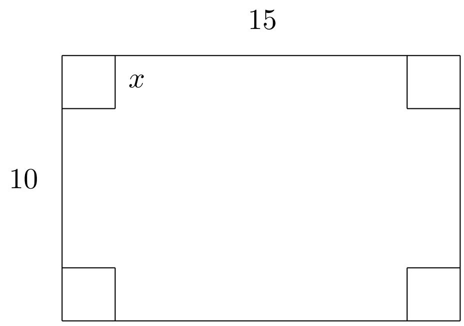
$$h = \frac{V}{\pi r^2}, \qquad \text{and so} \qquad S = 2\pi r^2 + \frac{2V}{r}.$$

Now

$$\frac{\mathrm{d}S}{\mathrm{d}r} = 4\pi r - \frac{2V}{r^2} = 0 \qquad \text{when} \qquad 4\pi r^3 = 2V = 2\pi r^2 h,$$

giving $h/r = 2$, so that the height should be twice the radius.

EXAMPLE 5.3.3. A pan is to be made from a rectangular sheet of tin measuring 10 centimetres by 15 centimetres by cutting squares of side x centimetres at the four corners and turning up the tin to form the sides. We wish to maximize the volume of the pan.



Clearly the volume is given by $V(x) = x(15 - 2x)(10 - 2x) = 150x - 50x^2 + 4x^3$, so that

$$V'(x) = 150 - 100x + 12x^2 = 0 \qquad \text{when} \qquad x = \frac{25 \pm 5\sqrt{7}}{6}.$$

Here the larger solution has to be discarded, since clearly it exceeds 5, an impossibility. Hence we only retain the smaller solution.

Sometimes it may be quite awkward to express the quantity we wish to maximize or minimize in terms of only one variable, so perhaps we try to adopt an alternative strategy such as this: (1) Identify what we want to maximize or minimize. (2) Express the quantity z we wish to maximize or minimize as a function of more than one variable but subject to constraints. (3) Use implicit differentiation to study the problem by differentiating z with respect to one of the variables x, and study the stationary points. (4) Decide whether the maximum or minimum in question occurs at one of the solutions in the previous step. Sometimes, it is clear from the nature of the problem that a maximum or minimum exists, so the situation will be clear if there is only one solution arising from the previous step. If there are more solutions in the previous step, then some extra care needs to be exercised.

EXAMPLE 5.3.4. We return to our problem in Example 5.3.1 where we wish to find two real numbers with sum 10 and with product as large as possible. If we denote the two numbers by x and y, then these two variables are constrained by the equation $x + y = 10$, since their sum is 10. We now wish to maximize the product $z = xy$ under this constraint. Using implicit differentiation, we have

$$\frac{\mathrm{d}z}{\mathrm{d}x} = y + x\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \qquad \text{when} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x}.$$

However, the constraint $x + y = 10$ gives, on differentiating with respect to x,

$$1 + \frac{\mathrm{d}y}{\mathrm{d}x} = 0, \qquad \text{so that} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = -1.$$

The stationary point occurs when $y/x = 1$, so that $x = y$. Combining this with the constraint $x+y = 10$, we obtain $x = y = 5$, with product $z = xy = 25$ as before.

EXAMPLE 5.3.5. We return to our problem in Example 5.3.2, where we wish to find the ratio between the height h and the radius r of the base of a can with fixed volume V so as to minimize the amount of metal used, with the understanding that the side and ends are made of the same metal. Here the radius r and the height h are subject to the constraint $\pi r^2 h = V$, where V is a constant. We wish to minimize the surface area $S = 2\pi r^2 + 2\pi rh$. Using implicit differentiation, we have

$$\frac{\mathrm{d}S}{\mathrm{d}r} = 4\pi r + 2\pi h + 2\pi r\frac{\mathrm{d}h}{\mathrm{d}r} = 0 \qquad \text{when} \qquad \frac{\mathrm{d}h}{\mathrm{d}r} = -\frac{2r + h}{r}.$$

However, the constraint $\pi r^2 h = V$ gives, on differentiating with respect to r,

$$2h + r\frac{\mathrm{d}h}{\mathrm{d}r} = 0, \qquad \text{so that} \qquad \frac{\mathrm{d}h}{\mathrm{d}r} = -\frac{2h}{r}.$$

The stationary point occurs when $2r + h = 2h$, so that $h/r = 2$, and so the height should be twice the radius as before.

EXAMPLE 5.3.6. Suppose that we have 16 metres of fencing to fence off a rectangular area. What is the largest area that can be fenced off? To do this, let x and y denote the lengths (in metres) of two adjacent sides of the rectangle, and let z denote the area of the rectangle. Then we have to maximize the value of

$$z = xy \tag{2}$$

subject to the constraint (of the length of the fence)

$$2x + 2y = 16. \tag{3}$$

Differentiating (2) and (3) with respect to x, we obtain respectively

$$\frac{\mathrm{d}z}{\mathrm{d}x} = y + x\frac{\mathrm{d}y}{\mathrm{d}x}$$

and

$$2 + 2\frac{\mathrm{d}y}{\mathrm{d}x} = 0. \tag{4}$$

When z is maximized, we must have $\mathrm{d}z/\mathrm{d}x = 0$, so that

$$y + x\frac{\mathrm{d}y}{\mathrm{d}x} = 0. \tag{5}$$

Combining (4) and (5) and eliminating $\mathrm{d}y/\mathrm{d}x$, we obtain

$$x = y. \tag{6}$$

Combining (3) and (6), we obtain $x = 4$. It follows that the maximum value of z is 16. Can you also find the minimum value of z?

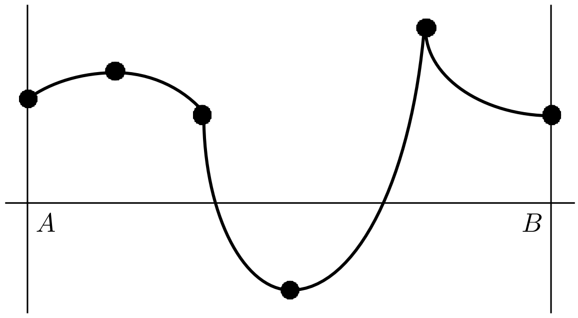
5.4. Global Maxima and Minima

DEFINITION. Suppose that $f(x)$ is a function defined on an interval I in $\mathbb{R}$.
(1) We say that $f(x)$ has a global minimum at $x_1 \in I$ if $f(x) \geq f(x_1)$ for every $x \in I$.
(2) We say that $f(x)$ has a global maximum at $x_2 \in I$ if $f(x) \leq f(x_2)$ for every $x \in I$.

To find global maximum and minimum points of a continuous function $f(x)$ in a closed interval $[A, B]$, we need to compare values of the function at the following points:

- any point $x \in [A, B]$ such that $f'(x) = 0$ – these are the stationary points;
- the endpoints $x = A$ and $x = B$; and
- any point $a \in [A, B]$ such that $f(x)$ is not differentiable at $x = a$.

The picture below illustrates our point.



To find global maximum and minimum points, if they exist, of a continuous function $f(x)$ in an open interval (A, B), we need to compare values of the function at the following points:

- any point $x \in [A, B]$ such that $f'(x) = 0$ – these are the stationary points;
- any point $a \in [A, B]$ such that $f(x)$ is not differentiable at $x = a$; and

check very carefully the behaviour of the function $f(x)$ as $x \to A$ from the right and as $x \to B$ from the left.

EXAMPLE 5.4.1. Consider the function $f(x) = x$ in the open interval $(0, 1)$. We have $f'(x) \neq 0$ for any $x \in (0, 1)$, and $f(x)$ is differentiable at every $x \in (0, 1)$, so we concentrate on the behaviour of $f(x)$ as $x \to 0$ from the right and as $x \to 1$ from the left. Indeed, no point $a \in (0, 1)$ can give rise to a global maximum or minimum, since we can find $x \in (0, a)$ so that $f(x) < f(a)$, and we can find $x \in (a, 1)$ so that $f(x) > f(a)$.

EXAMPLE 5.4.2. Consider the function $f(x) = |x|$ in the open interval $(-1, 1)$. We have $f'(x) \neq 0$ for any $x \in (-1, 0)$ and any $x \in (0, 1)$, and $f(x)$ is not differentiable at $x = 0$. It is not difficult to note that $x = 0$ gives rise to a global minimum, since $f(0) = 0 \leq |x| = f(x)$ for every $x \in (-1, 1)$. On the other hand, no point $a \in (-1, 1)$ can give rise to a global maximum, since we can find $x \in (|a|, 1)$ so that $f(x) = |x| > |a| = f(a)$.

EXAMPLE 5.4.3. Consider the function $f(x) = \sin x$ in the open interval $(0, 2\pi)$. It is easy to see that $f'(x) = \cos x = 0$ at the stationary points $x = \frac{1}{2}\pi$ and $x = \frac{3}{2}\pi$, and that the function is differentiable everywhere in $(0, 2\pi)$. Note that $f(x) \to 0$ as the two endpoints, and since $f(\frac{1}{2}\pi) = 1$ and $f(\frac{3}{2}\pi) = -1$, any global maximum or minimum is not going to come near the endpoints of the interval $(0, 2\pi)$. Indeed, we have $f(\frac{3}{2}\pi) \leq f(x) \leq f(\frac{1}{2}\pi)$ for every $x \in (0, 2\pi)$, so that the function has a global minimum at $x = \frac{3}{2}\pi$ and a global maximum at $x = \frac{1}{2}\pi$.

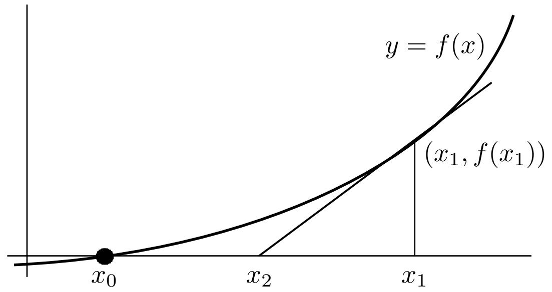
5.5. Newton's Method

In this section, we give an intuitive discussion of Newton's method, starting with an approximation x_1 of a solution of an equation $f(x) = 0$ and successively obtaining numbers $x_2, x_3, x_4, \ldots$ which, under appropriate conditions, are better and better approximations of a solution. Here we assume that the derivative $f'(x)$ exists in a range of values of x under consideration. We further assume that $f'(x) \neq 0$ at any point $x \in \mathbb{R}$ where we wish to use it.

We start with an approximation x_1 of a solution x_0 of the equation $f(x) = 0$, where $f'(x_1) \neq 0$. Then the tangent to the curve at the point $(x_1, f(x_1))$ has slope $f'(x_1)$ and intersects the horizontal axis at a unique point x_2. Clearly

$$\frac{f(x_1) - 0}{x_1 - x_2} = f'(x_1), \qquad \text{so that} \qquad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)},$$

as illustrated in the picture below.



We then repeat the argument, starting with the approximation x_2 of a solution of the equation $f(x) = 0$, and obtain another approximation

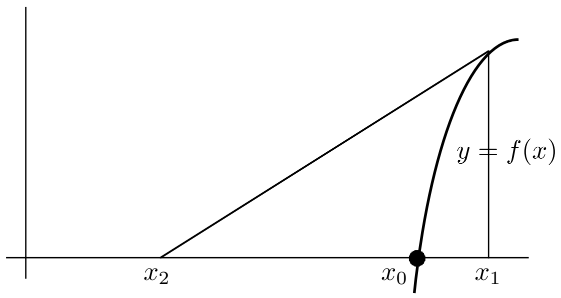
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}.$$

And so on. Then for every $n \in \mathbb{N}$, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

provided that the argument does not break down. The numbers $x_1, x_2, x_3, \ldots, x_n, \ldots$ are called the Newton iterates.

We do not give here conditions that are sufficient to ensure that the sequence $x_1, x_2, x_3, \ldots, x_n, \ldots$ moves towards a solution x_0 of the equation $f(x) = 0$. These and related questions are normally discussed in a course on numerical analysis. Instead, we give some examples on how the technique works. We begin by giving a graphical illustration of a worse approximation x_2 from the original approximation x_1 using Newton's method.



EXAMPLE 5.5.1. Let $f(x) = 2\cos x - x^2$. We apply Newton's method, starting with an approximation $x_1 = \frac{1}{3}\pi$ of a solution of the equation $f(x) = 0$. Note that $f'(x) = -2(\sin x + x)$. Since $x_1 = \frac{1}{3}\pi \approx 1.0472$, we obtain

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 1.0219 \qquad \text{and} \qquad x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 1.0217.$$

We remark that we could have started with the approximation $x_1 = \frac{1}{2}\pi$, but this would not have been so good. The problem originates from finding roots of the equation $2\cos x = x^2$. A reasonable sketch of the two functions $y = 2\cos x$ and $y = x^2$ in the same picture will show that $x = \frac{1}{3}\pi$ is much closer than $x = \frac{1}{2}\pi$ to the desired real number x_0 which is the first coordinate of the point of intersection of the two graphs.

EXAMPLE 5.5.2. We wish to find good approximations of $\sqrt{A}$, where A is a fixed positive real number. To do this, we write $f(x) = x^2 - A$, and try to find approximations of solutions of the equation $f(x) = 0$. Note that $f'(x) = 2x$, so that

$$x_2 = x_1 - \frac{x_1^2 - A}{2x_1} = \frac{1}{2}\left(x_1 + \frac{A}{x_1}\right), \qquad x_3 = \frac{1}{2}\left(x_2 + \frac{A}{x_2}\right),$$

and so on. Consider the case $A = 3$, starting with a first approximation $x_1 = 2$. Then

$$x_2 = \frac{1}{2}\left(x_1 + \frac{3}{x_1}\right) = 1.75, \qquad x_3 = \frac{1}{2}\left(x_2 + \frac{3}{x_2}\right) \approx 1.7321, \qquad x_4 = \frac{1}{2}\left(x_3 + \frac{3}{x_3}\right) \approx 1.7321,$$

not too far from the correct value of $\sqrt{3}$.

Problems for Chapter 5

1. Consider motion described by the displacement function $x(t) = t^3 - 4t^2 + 4t$.
   a) Sketch a graph of displacement against time, indicating clearly where the graph crosses the axes.
   b) Find $\dot{x}$ and $\ddot{x}$ as functions of time, and sketch their graphs against time.
   c) Determine carefully when the particle is (i) at the origin; (ii) to the right of the origin; (iii) to the left of the origin.
   d) Determine carefully when the particle is (i) stationary; (ii) moving to the right; (iii) moving to the left.
   e) Determine carefully when the particle is accelerating (i) to the right; (ii) to the left.

2. Consider motion described by the displacement function $x(t) = t^3 + 6t^2 - 15t + 8 = (t-1)^2(t+8)$.
   a) Sketch a graph of displacement against time, indicating clearly where the graph crosses the axes.
   b) Find $\dot{x}$ and $\ddot{x}$ as functions of time, and sketch their graphs against time.
   c) Determine carefully when the particle is (i) at the origin; (ii) to the right of the origin; (iii) to the left of the origin.
   d) Determine carefully when the particle is (i) stationary; (ii) moving to the right; (iii) moving to the left.
   e) Determine carefully when the particle is accelerating (i) to the right; (ii) to the left.

3. Show that each of the following displacement functions represents simple harmonic motion (taken to include those described in Example 5.1.5), and give the amplitude and period:
   a) $x(t) = \cos t + \sin t$
   b) $x(t) = \cos 2t - \sin 2t$
   c) $x(t) = 12\cos t - 5\sin t$
   d) $x(t) = 3\cos 2\pi t + 4\sin 2\pi t$
   e) $x(t) = \cos^2 t$
   f) $x(t) = \sin^2 2\pi t$
   g) $x(t) = \cos^2 4t - \sin^2 4t$
   h) $x(t) = 2\sin 6t \cos 6t$

4. A compact disc manufacturer works with a cost function $C(x) = 90000 + 500x + \frac{1}{100}x^2$ and a revenue function $R(x) = 1000x - \frac{1}{20}x^2$, where x is the number of hundreds of compact discs manufactured and sold.
   a) How many compact discs should be produced to minimize the average cost?
   b) What is the minimum average cost per hundred compact discs?
   c) Determine the marginal profit.
   d) How many compact discs should be produced and sold to maximize profit?

5. A company sells 9000 chairs per year. It has a carrying cost of 50 cents per year for each unsold chair stored. The company also reorders stock in fixed sized lots when it runs out of stock, and any such order is delivered immediately. The cost for each lot ordered is 25 cents per chair and 10 dollars for the paperwork. Let x be the fixed size of each order, so that the average number of unsold chairs over the year is $\frac{1}{2}x$.
   a) Show that the annual cost $C(x)$ in dollars of ordering x chairs at a time and storage of unsold chairs is given by $C(x) = 2250 + \frac{1}{4}x + 90000x^{-1}$.
   b) How many orders should be made each year in order to minimize the annual cost?

6. a) A consultant offers his services at 1000 dollars per client. If there are at least 100 clients, then the consultant offers every client a discount of 5 dollars for each additional client beyond the first 100. Furthermore, the consultant has the capacity to offer his service to a maximum of 250 clients.
      (i) What is the number of clients that maximizes the revenue for the consultant?
      (ii) What is the fee for each client in this situation?
   b) Suppose that the cost to the consultant is made up of an initial cost of 40000 dollars and then 200 dollars per client.
      (i) What is the maximum profit for the consultant?
      (ii) How many clients must the consultant have for this maximum to be reached?
      (iii) What is the fee for each client in this situation?

7. The wholesale price for each camera is 390 dollars. The business also has a monthly overhead of 1100 dollars. It is known that if the sale price of each camera is p dollars, then the number x of sales each month is given by $x = 50 - \frac{1}{20}p$. Determine the retail price, to the nearest dollar, for each camera in order to maximize the profit.

8. The cost in dollars to buy and distribute x cans of fruits is given by $C(x) = 65 + \frac{3}{100}x + \frac{1}{40000}x^2$. It is also known that the number x of cans that can be sold per day is related to the price of p dollars per can by $p = 23 - \frac{1}{400}x$.
   a) Find the price per can that will give maximum profit.
   b) How many cans are sold daily in tis situation?

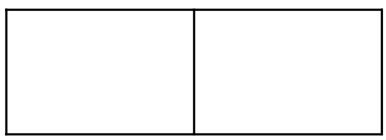
9. Find two positive real numbers x and y such that $x + y = 6$ and xy^2 is as large as possible.

10. Find the point on the parabola $y = x^2$ which is closest to the point $(6, 3)$.

11. A box of volume 108 cubic centimetres having a square base and open top is to be constructed. What dimension of the box will minimize the amount of material used?

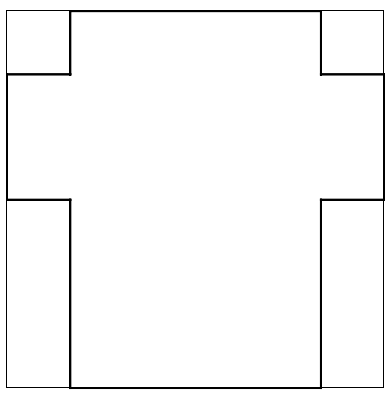
12. An open box with square base and of volume 96 cubic centimetres is to be constructed. The material for the base costs three times as much as that material for the sides. What dimension of the box will minimize the cost?

13. We have 1200 metres of fencing to enclose a twin paddock with two rectangilar regions of equal area as shown.



Find the maximum area that can be enclosed.

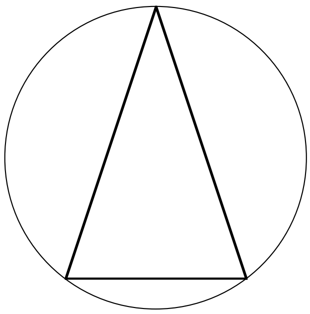
14. A box with lid is to be made from a square sheet of tin with side length 20 centimetres. The sheet is cut along the lines indicated, then turned up to form the ends and sides, and the flap is turned over to form the lid.



Find the maximum volume of the box.

15. A rectangular box of fixed volume V is to be twice as long as it is wide. The material for the top and the four sides costs three times as much as the material for the base. Find the proportion of the box that will minimize the cost.

16. Find the area of the largest isosceles triangle that can be inscribed in a circle of radius R.



17. Show that for all rectangles with constant perimeter p, the square has the greatest area.

18. Find the dimensions of the cylindrical can of total surface area $S = 150\pi$ square centimetres, including the base and the lid, and for which the volume is a maximum.

19. A cylindrical barrel is to be constructed to hold 256π cubic metres of liquid. The cost per square metre of constructing the side of the barrel is half the cost per square metre of constructing the top and the bottom. What are the dimensions of the barrel that costs the least to construct?

20. Find the maximum volume of a cylinder inscribed inside a sphere of radius R.

21. Find the global maximum and minimum points, if they exist, of the function $f(x) = x^3 - 9x^2 - 48x + 3$ on each of the following intervals:
a) $[-5, 12]$ b) $[-4, 17]$ c) $[-10, 12]$ d) $[-10, 17]$
e) $(0, \infty)$ f) $(-\infty, 0)$ g) $(-\infty, -5)$ h) $(14, \infty)$

22. Find the global maximum and minimum points, if they exist, of each of the following functions in the open interval $(-\infty, \infty)$:
a) $f(x) = 4x - x^2 + 5$ b) $f(x) = \dfrac{x}{1 + x^2}$ c) $f(x) = (2 + \cos^2 x) \sin x$

23. Find the global maximum and minimum points, if they exist, of each of the following functions in the open interval $(0, \infty)$:
a) $f(x) = x\mathrm{e}^{-x}$ b) $f(x) = x + \dfrac{1}{x}$ c) $f(x) = x - \log x$

24. For each of the following functions $f(x)$, make a first approximation x_1 of a solution of the equation $f(x) = 0$, and compute Newton iterates until two sucessive iterates agree to two decimal places:
a) $f(x) = 3x^2 - 4x - 5$ b) $f(x) = x - 1 + \sin x$ c) $f(x) = \dfrac{1}{x^2 + 4} - 3x + 4$

25. Find an iterative process for computing $\sqrt[3]{A}$ where A is a given positive real number.