FIRST YEAR CALCULUS

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Chapter 6

LIMITS OF FUNCTIONS

6.1. Introduction

We study the problem of the behaviour of a real valued function $f(x)$ as the real variable x gets close to a given real number a, and begin by looking at a few simple examples.

EXAMPLE 6.1.1. Consider the function $f(x) = x^3 + x$. Let us study its behaviour as x gets close to the real number 1, but is not equal to 1. We have the following numerical data:

$$f(1.1) = 2.431, \qquad f(1.01) = 2.040301, \qquad f(1.001) = 2.004003001,$$
$$f(0.9) = 1.629, \qquad f(0.99) = 1.960299, \qquad f(0.999) = 1.996002999.$$

From this limited evidence, we suspect that $f(x)$ is close to the value 2 when x is close to 1. Note here also that $f(1) = 2$. We would therefore like to say that

$$\lim_{x\to 1} f(x) = 2 = f(1).$$

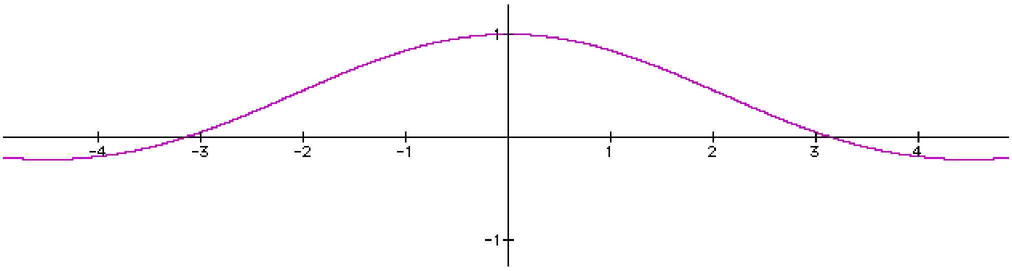
EXAMPLE 6.1.2. Consider the function $f(x) = (x^3 - 1)/(x - 1)$. Let us study its behaviour as x gets close to the real number 1, but is not equal to 1. We have the following numerical data:

$$f(1.1) = 3.31, \qquad f(1.01) = 3.0301, \qquad f(1.001) = 3.003001,$$
$$f(0.9) = 2.71, \qquad f(0.99) = 2.9701, \qquad f(0.999) = 2.997001.$$

From this limited evidence, we suspect that $f(x)$ is close to the value 3 when x is close to 1. While the function $f(x)$ is not defined at $x = 1$, we would nevertheless like to say that

$$\lim_{x\to 1} f(x) = 3.$$

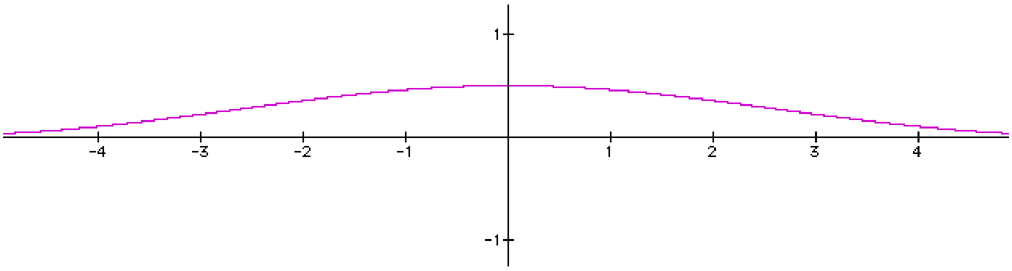
Example 6.1.3. Consider the function $f(x) = x^{-1} \sin x$. Let us study its behaviour as = x gets close to the real number 0, but is not equal to 0.



From the graph, we suspect that $f(x)$ is close to the value 1 when x is close to 0. While the function $f(x)$ is not defined at $x = 0$, we would nevertheless like to say that

$$\lim_{x \to 0} f(x) = 1.$$

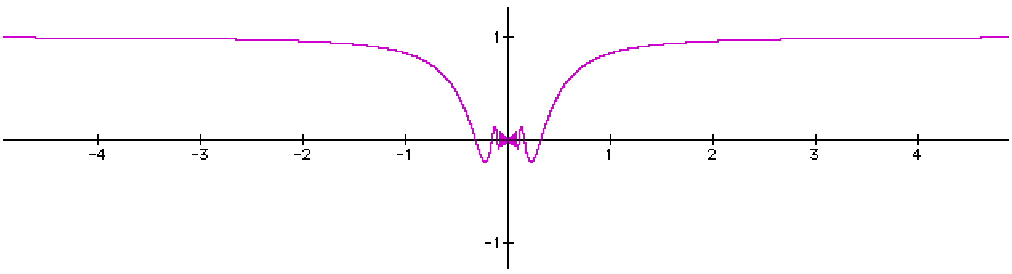
Example 6.1.4. Consider the function $f(x) = x^{-2}(1 - \cos x)$. Let us study its behaviour as x gets close to the real number 0, but is not equal to 0.



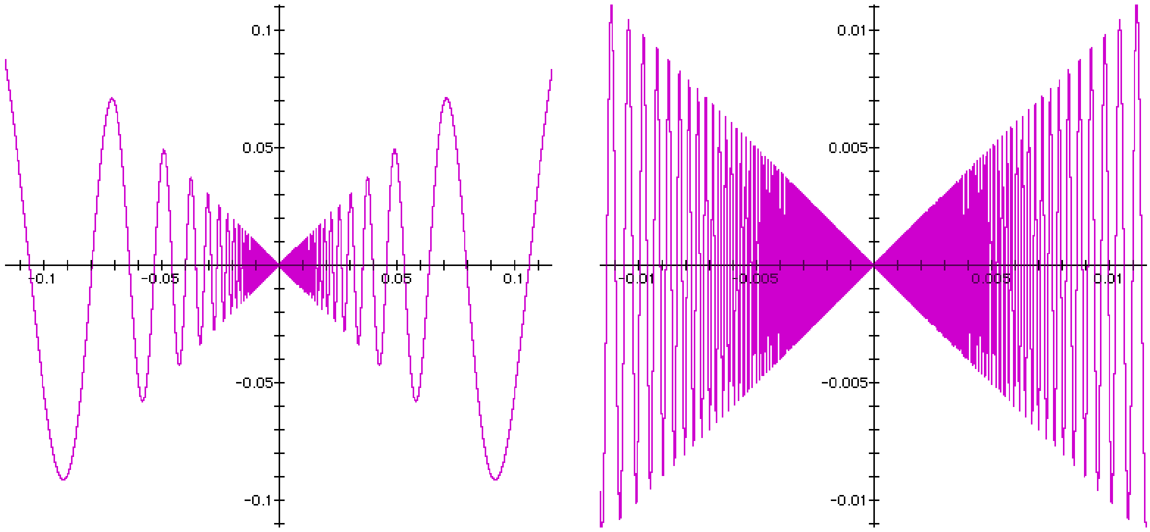
From the graph, we suspect that $f(x)$ is close to the value $\frac{1}{2}$ when x is close to 0. While the function $f(x)$ is not defined at $x = 0$, we would nevertheless like to say that

$$\lim_{x \to 0} f(x) = \frac{1}{2}.$$

Example 6.1.5. Consider the function $f(x) = x \sin(1/x)$. Let us study its behaviour as x gets close to the real number 0, but is not equal to 0.



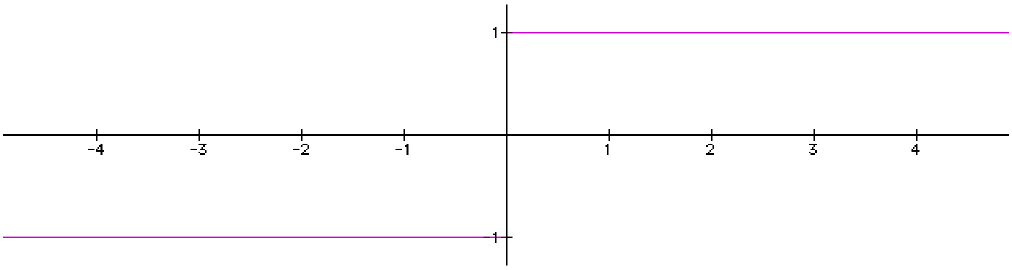
It appears that $f(x)$ is close to the value 0 when x is close to 0. Let us look more closely.



While the function $f(x)$ is not defined at $x = 0$, we would nevertheless like to say that

$$\lim_{x \to 0} f(x) = 0.$$

EXAMPLE 6.1.6. Consider the function $f(x) = x/|x|$. Let us study its behaviour as x gets close to the real number 0, but is not equal to 0. Clearly $f(x) = 1$ when $x > 0$ and $f(x) = -1$ when $x < 0$.



It follows that when x is close to 0, but not equal to 0, then $f(x)$ is close to the value 1 or close to the value -1, depending on whether x is positive or negative. It is therefore clear that $f(x)$ has no limit as $x \to 0$. On the other hand, it is reasonable to say that $f(x)$ is close to the value 1 when $x > 0$ is close to 0, and that $f(x)$ is close to the value -1 when $x < 0$ is close to 0. In this case, we would like to say that

$$\lim_{x \to 1} f(x)$$

does not exist, but also that

$$\lim_{\substack{x \to 0 \\ x > 0}} f(x) = 1 \qquad \text{and} \qquad \lim_{\substack{x \to 0 \\ x < 0}} f(x) = -1.$$

In order to formulate a proper definition for a limit, we need to study the differences

$$|x - a| \qquad \text{and} \qquad |f(x) - L|,$$

and find suitable ways to describe their smallness. In order to conclude that $f(x) \to L$ as $x \to a$, we must therefore be able to convince ourselves that to make $|f(x) - L|$ small, it is sufficient to make $|x - a|$ small enough.

Definition. We say that $f(x) \to L$ as $x \to a$, or

$$\lim_{x\to a} f(x) = L,$$

if, for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

Remark. Note that we omit discussion of the case $x = 1$ in Example 6.1.2 and the case $x = 0$ in Examples 6.1.3–6.1.6. After all, we are only interested in those values of x which are close to a but not equal to a. The purpose of the restriction $|x - a| > 0$ is to omit discussion of the case when $x = a$.

Example 6.1.7. Consider the function $f(x) = 2x + 3$. Let us study its behaviour as $x \to 1$. Of course, we suspect that $f(x) \to 5$ as $x \to 1$. Here $a = 1$ and $L = 5$. We therefore need to study the differences $|x - 1|$ and $|f(x) - 5|$. Let $\epsilon > 0$ be chosen. Then

$$|f(x) - 5| = |2x + 3 - 5| = |2x - 2| = 2|x - 1| < \epsilon$$

whenever $|x - 1| < \delta = \epsilon/2$.

Example 6.1.8. Consider the function $f(x) = x^2$. Let us study its behaviour as $x \to 0$. Of course, we suspect that $f(x) \to 0$ as $x \to 0$. Here $a = 0$ and $L = 0$. We therefore need to study the differences $|x - 0|$ and $|f(x) - 0|$. Let $\epsilon > 0$ be chosen. Then

$$|f(x) - 0| = |x^2| < \epsilon$$

whenever $|x - 0| = |x| < \delta = \sqrt{\epsilon}$.

Example 6.1.9. Let us return to Example 6.1.1, and consider again the function $f(x) = x^3 + x$ when $x \to 1$. We would like to show that $f(x) \to 2$ as $x \to 1$. Here $a = 1$ and $L = 2$. We therefore need to study the differences $|x - 1|$ and $|f(x) - 2|$. Let $\epsilon > 0$ be chosen. Then

$$|f(x) - 2| = |x^3 + x - 2| \le |x^3 - 1| + |x - 1| = |x^2 + x + 1||x - 1| + |x - 1|.$$

Since we are only interested in those values of x close to 1, we shall lose nothing by considering only those values of x satisfying $0 < x < 2$. Then $|x^2 + x + 1| = x^2 + x + 1 < 7$. It follows that if $0 < x < 2$, then

$$|f(x) - 2| < 8|x - 1| < \epsilon$$

if we have the additional restriction $|x - 1| < \epsilon/8$. Note now that $|x - 1| < 1$ will guarantee $0 < x < 2$. Hence $|f(x) - 2| < \epsilon$ can be guaranteed by $|x - 1| < \min\{1, \epsilon/8\}$. It follows that the requirements of the definition are satisfied if we take $\delta = \min\{1, \epsilon/8\}$.

Remark. The choice of δ is by no means unique. Suppose that in Example 6.1.9, we restrict our attention only to those values of x satisfying $0 < x < 1.5$. Then $|x^2 + x + 1| = x^2 + x + 1 < 5$. It follows that if $0 < x < 1.5$, then

$$|f(x) - 2| < 6|x - 1| < \epsilon$$

if we have the additional restriction $|x - 1| < \epsilon/6$. Note now that $|x - 1| < 0.5$ will guarantee $0 < x < 1.5$. Hence $|f(x) - 2| < \epsilon$ can be guaranteed by $|x - 1| < \min\{0.5, \epsilon/6\}$. It follows that the requirements of the definition are satisfied also if we take $\delta = \min\{0.5, \epsilon/6\}$. Indeed, in many situations, it will be very difficult, if not impossible, to obtain the best possible choice of δ. We are only interested in finding one value of δ that satisfies the requirements. Whether it is best possible or not is not important.

6.2. Further Techniques

The techniques of Examples 6.1.7–6.1.9 may be useful only in simple cases. If the given function is somewhat complicated, then the same approach will at best lead to a very complicated argument. An alternative is to seek ways to split the given function into "smaller" manageable parts. As an illustration, consider the function $f(x) = x^3 + x$ discussed in Example 6.1.9. We may choose to study the functions x^3 and x separately, and note that the function x^3 is the product of three copies of the function x.

The following result is called the Arithmetic of limits, comprising respectively the sum, product and quotient rules.

PROPOSITION 6A. *Suppose that the functions $f(x) \to L$ and $g(x) \to M$ as $x \to a$. Then*
(a) $f(x) + g(x) \to L + M$ as $x \to a$;
(b) $f(x)g(x) \to LM$ as $x \to a$; and
(c) if $M \neq 0$, then $f(x)/g(x) \to L/M$ as $x \to a$.

PROOF. (a) We shall use the inequality

$$|(f(x) + g(x)) - (L + M)| \leq |f(x) - L| + |g(x) - M|.$$

Given any $\epsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$|f(x) - L| < \epsilon/2 \quad \text{whenever } 0 < |x - a| < \delta_1,$$

and

$$|g(x) - M| < \epsilon/2 \quad \text{whenever } 0 < |x - a| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\} > 0$. It follows that whenever $0 < |x - a| < \delta$, we have

$$|(f(x) + g(x)) - (L + M)| \leq |f(x) - L| + |g(x) - M| < \epsilon.$$

(b) We shall use the inequality

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - f(x)M + f(x)M - LM| \\ &= |f(x)(g(x) - M) + (f(x) - L)M| \\ &\leq |f(x)||g(x) - M| + |M||f(x) - L|. \end{aligned}$$

Since $f(x) \to L$ as $x \to a$, there exists $\delta_1 > 0$ such that

$$|f(x) - L| < 1 \quad \text{whenever } 0 < |x - a| < \delta_1,$$

so that

$$|f(x)| < |L| + 1 \quad \text{whenever } 0 < |x - a| < \delta_1.$$

On the other hand, given any $\epsilon > 0$, there exist $\delta_2, \delta_3 > 0$ such that

$$|f(x) - L| < \frac{\epsilon}{2(|M| + 1)} \quad \text{whenever } 0 < |x - a| < \delta_2,$$

and

$$|g(x) - M| < \frac{\epsilon}{2(|L| + 1)} \quad \text{whenever } 0 < |x - a| < \delta_3.$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\} > 0$. It follows that whenever $0 < |x - a| < \delta$, we have

$$|f(x)g(x) - LM| \leq |f(x)||g(x) - M| + |M||f(x) - L| < \epsilon.$$

(c) We shall first show that $1/g(x) \to 1/M$ as $x \to a$. To do this, we shall use the identity

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \frac{|g(x) - M|}{|g(x)||M|}.$$

Since $M \neq 0$ and $g(x) \to M$ as $x \to a$, there exists $\delta_1 > 0$ such that

$$|g(x) - M| < |M|/2 \quad \text{whenever } 0 < |x - a| < \delta_1,$$

so that

$$|g(x)| > |M|/2 \quad \text{whenever } 0 < |x - a| < \delta_1.$$

On the other hand, given any $\epsilon > 0$, there exists $\delta_2 > 0$ such that

$$|g(x) - M| < M^2\epsilon/2 \quad \text{whenever } 0 < |x - a| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\} > 0$. It follows that whenever $0 < |x - a| < \delta$, we have

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \frac{|g(x) - M|}{|g(x)||M|} \leq \frac{2|g(x) - M|}{|M|^2} < \epsilon.$$

We now apply part (b) to $f(x)$ and $1/g(x)$ to get the desired result. ○

Remark. Note that for the quotient rule, we must impose the restriction that $M \neq 0$. Division by 0 is meaningless.

Example 6.2.1. Consider the function

$$h(x) = \frac{2x^3 + 5x + 3}{x^3 + 3x^2 + 1}$$

as $x \to 2$. Clearly we have $x^2 \to 4$, $x^3 \to 8$. On the other hand, the constant function $2 \to 2$, so that the function $2x^3$, being the product of the constant function 2 and the function x^3, satisfies $2x^3 \to 16$ by the product rule. Similarly, we have $5x \to 10$ and $3x^2 \to 12$. Naturally $3 \to 3$ and $1 \to 1$. It follows that as $x \to 2$, we have

$$h(x) = \frac{2x^3 + 5x + 3}{x^3 + 3x^2 + 1} \to \frac{16 + 10 + 3}{8 + 12 + 1} = \frac{29}{21}.$$

Example 6.2.2. Consider the function

$$h(x) = \frac{\sin x + \cos x}{\sin x - 2\cos x}$$

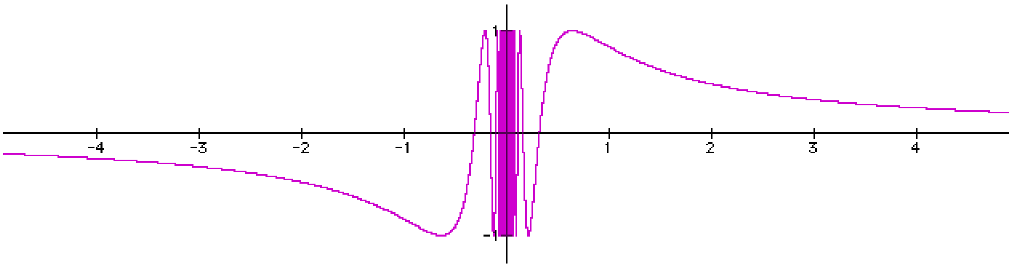
as $x \to \pi/2$. Here, we assume knowledge that $\sin x \to 1$ and $\cos x \to 0$ as $x \to \pi/2$. Then clearly, as $x \to \pi/2$, we have

$$h(x) = \frac{\sin x + \cos x}{\sin x - 2\cos x} \to \frac{1 + 0}{1 - 0} = 1.$$

A second alternative that we may pursue is to squeeze a given function between two known functions that have the same limit. As an illustration, consider the function $f(x) = x\sin x$. Since $-1 \leq \sin x \leq 1$ always, we have $-|x| \leq f(x) \leq |x|$. As $x \to 0$, we clearly have $|x| \to 0$. But then the function $f(x)$ is squeezed between $|x|$ and $-|x|$ which both converge to 0.

PROPOSITION 6B. (SQUEEZING PRINCIPLE) *Suppose that $g(x) \le f(x) \le h(x)$ for every $x \ne a$ in some open interval that contains a. Suppose further that $g(x) \to L$ and $h(x) \to L$ as $x \to a$. Then $f(x) \to L$ as $x \to a$.*

REMARK. It is crucial that squeezing occurs, in that $g(x)$ and $h(x)$ go to the same limit. To see that this is necessary, we use the well known result (see Problem 10) that the function $f(x) = \sin(1/x)$ does not approach a limit as $x \to 0$. Clearly $-1 \le f(x) \le 1$, but squeezing does not occur.



PROOF OF PROPOSITION 6B. By Proposition 6A, we have $h(x) - g(x) \to 0$ as $x \to a$. We shall use the inequality

$$|f(x) - L| = |(f(x) - g(x)) + (g(x) - L)| \le |f(x) - g(x)| + |g(x) - L| \le |h(x) - g(x)| + |g(x) - L|.$$

Given any $\epsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$|h(x) - g(x)| < \epsilon/2 \quad \text{whenever } 0 < |x - a| < \delta_1,$$

and

$$|g(x) - L| < \epsilon/2 \quad \text{whenever } 0 < |x - a| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\} > 0$. It follows that whenever $0 < |x - a| < \delta$, we have

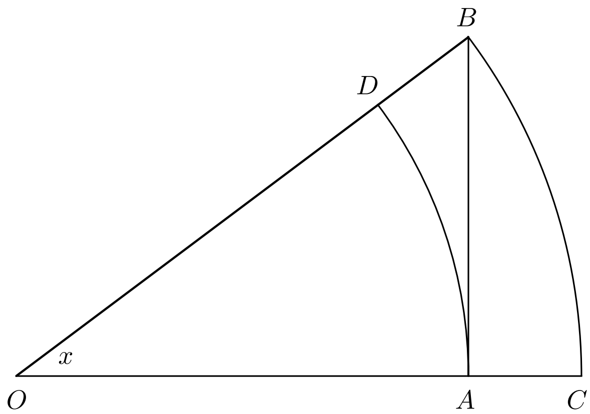
$$|f(x) - L| \le |h(x) - g(x)| + |g(x) - L| < \epsilon$$

as required. ◯

EXAMPLE 6.2.3. We shall show that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1. \qquad (1)$$

To do this, we shall use some very simple geometric ideas to find two functions $g(x)$ and $h(x)$ to squeeze together. Suppose first of all that $0 < x < \pi/2$.



Let OAB be a right angled triangle formed by the points $O(0,0)$, $A(\cos x, 0)$ and $B(\cos x, \sin x)$. Note then that the angle AOB has value x in radians. Note also that the points B and $C(1,0)$ both lie on the circle of radius 1 and centred at O. Finally, let D be the intersection point of the segment OB with the circle passing through A and centred at O. Suppose that we write

$$\begin{aligned} \alpha &= \text{area of circular segment } OAD, \\ \beta &= \text{area of triangle } OAB, \\ \gamma &= \text{area of circular segment } OCB. \end{aligned}$$

Then clearly $\alpha < \beta < \gamma$. On the other hand, simple calculation gives $2\alpha = x\cos^2 x$, $2\beta = \sin x \cos x$ and $2\gamma = x$, so that

$$\cos x < \frac{\sin x}{x} < \frac{1}{\cos x}. \tag{2}$$

Note now that all the three terms in (2) remain unchanged if x is replaced by $-x$. It follows that (2) is valid for all $x \neq 0$ in the open interval $(-\pi/2, \pi/2)$. Now take $g(x) = \cos x$ and $h(x) = 1/\cos x$. Then clearly $g(x) \to 1$ and $h(x) \to 1$ as $x \to 0$. The assertion (1) now follows.

6.3. One Sided Limits

Recall Example 6.1.6, and consider also the following example.

Example 6.3.1. Consider the function

$$f(x) = \begin{cases} x+2 & \text{if } x > 3, \\ x+3 & \text{if } x \le 3. \end{cases}$$

Then it is not difficult to see that as $x \to 3$, the limit does not exist. On the other hand, it is easy to see that $f(x)$ is close to the value 5 when $x > 3$ is close to 3, and that $f(x)$ is close to the value 6 when $x < 3$ is close to 3. If we limit the approach to 3 to just from one side, then we can formulate one sided limits.

Definition. We say that $f(x) \to L$ as $x \to a+$, or

$$\lim_{x \to a+} f(x) = L,$$

if, for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < x - a < \delta$. In this case, L is called the right hand limit.

Definition. We say that $f(x) \to L$ as $x \to a-$, or

$$\lim_{x \to a-} f(x) = L,$$

if, for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < a - x < \delta$. In this case, L is called the left hand limit.

Example 6.3.2. Let us return to the function $f(x)$ in Example 6.3.1. We have

$$\lim_{x \to 3-} f(x) = 6 \qquad \text{and} \qquad \lim_{x \to 3+} f(x) = 5.$$

Example 6.3.3. Let us return to the function $f(x) = x/|x|$ in Example 6.1.6. We have

$$\lim_{x \to 0-} f(x) = -1 \qquad \text{and} \qquad \lim_{x \to 0+} f(x) = 1.$$

It is very easy to deduce the following result.

PROPOSITION 6C. *We have*

$$\lim_{x\to a} f(x) = L \qquad \textit{if and only if} \qquad \lim_{x\to a-} f(x) = \lim_{x\to a+} f(x) = L.$$

It is not difficult to formulate suitable analogues of the Arithmetic of limits and the Squeezing principle. Their precise statements are left as exercises.

6.4. Infinite Limits

Consider the function $f(x) = 1/x$ when $x \to 0$. Although $f(x)$ does not approach a finite limit, it is not difficult to accept that we can still say something about the behaviour of $f(x)$ when $x \to 0$, namely that $f(x)$ gets rather large.

DEFINITION. We say that a function $f(x)$ diverges to infinity, denoted by $f(x) \to \infty$ as $x \to a$, if, for every $E > 0$, there exists $\delta > 0$ such that $|f(x)| > E$ whenever $0 < |x - a| < \delta$.

EXAMPLE 6.4.1. Consider the function $f(x) = 1/x$. We suspect that $f(x) \to \infty$ as $x \to 0$. Here $a = 0$. Let $E > 0$ be chosen. Then

$$|f(x)| = |1/x| = 1/|x| > E$$

whenever $|x - 0| = |x| < \delta = 1/E$.

The following simple observation is useful.

PROPOSITION 6D. *The function $f(x) \to \infty$ as $x \to a$ if and only if the function $1/f(x) \to 0$ as $x \to a$.*

EXAMPLE 6.4.2. Consider the function $f(x) = 1/x \sin x$ as $x \to 0$. Let $g(x) = 1/f(x) = x \sin x$. We shall first of all show that $g(x) \to 0$ as $x \to 0$. Let $\epsilon > 0$ be given. Then

$$|g(x) - 0| = |x \sin x| \le |x| < \epsilon$$

whenever $0 < |x - 0| < \delta$ if we choose $\delta = \epsilon$. It now follows from Proposition 6D that $f(x) \to \infty$ as $x \to 0$.

REMARK. Note that the Arithmetic of limits in Section 6.2 does not extend to infinite limits. Consider, for example, $f(x) = 1/x$ and $g(x) = -1/x$. Then $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to 0$. Note, however, that $f(x) + g(x) \to 0$ as $x \to 0$.

6.5. Limits at Infinity

We now study the behaviour of a function $f(x)$ as $x \to +\infty$. The following definition is natural.

DEFINITION. We say that $f(x) \to L$ as $x \to +\infty$, or

$$\lim_{x\to+\infty} f(x) = L,$$

if, for every $\epsilon > 0$, there exists $D > 0$ such that $|f(x) - L| < \epsilon$ whenever $x > D$.

EXAMPLE 6.5.1. Consider the function $f(x) = 1/x^2$. Let us study its behaviour as $x \to +\infty$. Of course, we suspect that $f(x) \to 0$ as $x \to +\infty$. Here $L = 0$. To prove this, let $\epsilon > 0$ be chosen. Then

$$|f(x) - 0| = |1/x^2| = 1/x^2 < \epsilon \quad \text{whenever } x > D = \sqrt{\frac{1}{\epsilon}}.$$

We also study the behaviour of a function $f(x)$ as $x \to -\infty$. Corresponding to the above, we have the following obvious analogue.

DEFINITION. We say that $f(x) \to L$ as $x \to -\infty$, or

$$\lim_{x \to -\infty} f(x) = L,$$

if, for every $\epsilon > 0$, there exists $D > 0$ such that $|f(x) - L| < \epsilon$ whenever $x < -D$.

EXAMPLE 6.5.2. Consider the function $f(x) = 1 + x^{-1} \sin x$. Let us study its behaviour as $x \to -\infty$. Of course, we suspect that $f(x) \to 1$ as $x \to -\infty$. After all, we have $-1 \le \sin x \le 1$ always. Here $L = 1$. To prove this, let $\epsilon > 0$ be chosen. Then, for $x < 0$, we have

$$|f(x) - 1| = |x^{-1} \sin x| \le |x^{-1}| = -x^{-1} < \epsilon \quad \text{whenever } x < -D = -\frac{1}{\epsilon}.$$

[If you have difficulty following the calculation, note that if $a < b$ and $c < 0$, then $ac > bc$. Check the calculation again.]

Again, it is not difficult to formulate suitable analogues of the Arithmetic of limits and the Squeezing principle. Their precise statements are left as exercises.

Finally, we have the following extra definitions which we seldom use.

DEFINITION. We say that $f(x) \to \infty$ as $x \to +\infty$ if, for every $E > 0$, there exists $D > 0$ such that $|f(x)| > E$ whenever $x > D$.

DEFINITION. We say that $f(x) \to \infty$ as $x \to -\infty$ if, for every $E > 0$, there exists $D > 0$ such that $|f(x)| > E$ whenever $x < -D$.

Problems for Chapter 6

1. Use the definition of limit to prove each of the following:
   a) $\lim_{x\to 2}(4x+5)=13$ b) $\lim_{x\to 0}\sqrt{x}=0$

2. For each of the following functions, make a guess of the limit, then prove your assertion using the formal definition of a limit:
   a) $f(x)=3x+5$ as $x\to 2$ b) $f(x)=-4x+5$ as $x\to -1$
   c) $f(x)=x^2$ as $x\to 0$ d) $f(x)=|x-3|+1$ as $x\to 3$

3. Consider $f(x)=x^2$.
   a) Find a $\delta>0$ so that $|f(x)-4|<1/10$ when $|x-2|<\delta$.
   b) Use the formal definition of a limit to prove that $f(x)$ approaches 4 as x tends to 2.

4. a) Use the Arithmetic of limits to determine $\displaystyle\lim_{x\to -1}\frac{x^2+3x+2}{2x^2-8}$.
   b) By first factorizing the numerator and the denominator, determine $\displaystyle\lim_{x\to -2}\frac{x^2+3x+2}{2x^2-8}$. Explain all your steps carefully.

5. Use the identity $(a-b)(a+b)=a^2-b^2$ and the Arithmetic of limits to evaluate $\displaystyle\lim_{x\to 1}\frac{\sqrt{1+3x}-2}{x-1}$.

6. Use the Arithmetic of limits in a suitable way to evaluate $\displaystyle\lim_{x\to +\infty}\frac{3x^4+2x^3+5x^2+2}{4x^4+5x^2+x+3}$.

7. Use the Arithmetic of limits to determine each of the following, and explain each step carefully:
   a) $\displaystyle\lim_{x\to -1}\frac{2x^2+7x+5}{3x^2+5x+2}$ b) $\displaystyle\lim_{x\to \infty}\frac{x^2-5x+1}{3x^2-7x+2}$
   c) $\displaystyle\lim_{x\to +\infty}\sqrt{x^2-3x+1}-x$ d) $\displaystyle\lim_{x\to 0}\frac{1-\cos 4x}{x^2}$

   [Hint: In part (d), the fact that $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}=1$ may be useful. Do not try to use l'Hôpital's rule.]

8. Evaluate each of the following limits by using the Arithmetic of limits in a suitable way, and explain your steps carefully:
   a) $\displaystyle\lim_{x\to 2}\frac{x^2-4}{x-2}$ b) $\displaystyle\lim_{x\to 1}\frac{x^2-4x+3}{x^2-5x+4}$ c) $\displaystyle\lim_{x\to 1}\frac{x^3-1}{x^2-1}$
   d) $\displaystyle\lim_{x\to 2}\frac{\sqrt{x+7}-3}{x-2}$ e) $\displaystyle\lim_{x\to 1}\frac{\sqrt{1+3x}-2}{\sqrt{x+8}-3}$ f) $\displaystyle\lim_{x\to 1}\frac{x^{1/3}-1}{x-1}$

9. Use the Squeezing principle to find each of the following limits:
   a) $\displaystyle\lim_{x\to \infty}\frac{\cos 3x}{x}$ b) $\displaystyle\lim_{x\to 0}x^2\left(1+\cos\frac{1}{x}\right)$ c) $\displaystyle\lim_{x\to 0}x^2\sin\left(\frac{1}{x}\right)$
   d) $\displaystyle\lim_{x\to \infty}\frac{\cos x^2}{x}$ e) $\displaystyle\lim_{x\to 0}\sqrt{x}\sin x\cos x$

10. Consider the function $f(x)=\sin(1/x)$, defined for $x\neq 0$.
    a) Show that for every $\delta>0$, there exist $x_1,x_2\in(0,\delta)$ such that $f(x_1)=1$ and $f(x_2)=-1$.
    b) Show that for every real number $L\in\mathbb{R}$, we have $|f(x_1)-L|+|f(x_2)-L|\geq 2$, where x_1 and x_2 are the solutions in (a).
    c) Show that for every real number $L\in\mathbb{R}$ and every $\delta>0$, there exists $x_0\in(0,\delta)$ such that $|f(x_0)-L|\geq 1$.
    d) Explain why it is not true that $f(x)\to L$ as $x\to 0$.

11. The purpose of this problem is to prove that $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x}=0$. Follow carefully the steps indicated below:
    a) Let $f(x)=(1-\cos x)/x$. Convince yourself that $f(x)=-f(-x)$ for every non-zero $x\in\mathbb{R}$.
    b) Suppose first of all that $0<x<\pi/2$. Attempt to draw a diagram from the description below. Let OAB be a right angled triangle formed by the points $O(0,0)$, $A(\cos x,0)$ and $B(\cos x,\sin x)$, and note that the angle AOB has value x in radians. Note also that the points B and $C(1,0)$ both lie on the circle of radius 1 and centred at O. Using the fact that the length of the arc BC is greater than the length of the line segment BC, show that
    $$0<\frac{1-\cos x}{x}<\cos\left(\frac{\pi-x}{2}\right).$$
    c) Combining (a) and (b), deduce that for every real number x satisfying $0<|x|<\pi/2$, we have
    $$0<|f(x)|<\cos\left(\frac{\pi-|x|}{2}\right).$$
    d) Prove that $|f(x)|\to 0$ as $x\to 0$.
    e) Use the definition of limits to show that the result follows from (d).

12. Suppose that $f(x)\to L$ as $x\to a$. Prove that $|f(x)|\to|L|$ as $x\to a$.

13. Prove that $\displaystyle\lim_{x\to 0}\frac{(x^2+x)^{1/2}-x^{1/2}}{x^{3/2}}=\frac{1}{2}$.

14. Consider the function $f(x)=(1-\cos x)/x$.
    a) Show that for every $x\in\mathbb{R}$ satisfying $0<|x|<\pi/2$, we have $\displaystyle f(x)=\frac{\sin^2 x}{x(1+\cos x)}$.
    b) Using the Arithmetic of limits and the results $\cos x\to 1$ and $(\sin x)/x\to 1$ as $x\to 0$, show that $f(x)\to 0$ as $x\to 0$. You must explain each step carefully.
    c) Evaluate $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x^2}$. You must explain each step carefully.

15. Find each of the following limits:
    a) $\displaystyle\lim_{x\to 0}\frac{\sin(-5x)}{7x}$
    b) $\displaystyle\lim_{x\to 0}\frac{\sin 3x}{\tan(x/2)}$
    c) $\displaystyle\lim_{x\to 0}\frac{\cos x-1}{\sin^2 x}$

16. Evaluate each of the following limits by using the Arithmetic of limits in a suitable way, and explain your steps carefully:
    a) $\displaystyle\lim_{x\to+\infty}\frac{x+4}{x^2+x+5}$
    b) $\displaystyle\lim_{x\to+\infty}\frac{4x^2+x-6}{5x^2-x+10}$
    c) $\displaystyle\lim_{x\to+\infty}\frac{x^3+1}{x^2-1}$
    d) $\displaystyle\lim_{x\to+\infty}\left(\sqrt{x^2+4}-x\right)$
    e) $\displaystyle\lim_{x\to+\infty}\left(\sqrt{x^2+4x+3}-x\right)$
    f) $\displaystyle\lim_{x\to-\infty}\left(\sqrt{x^2+4x+3}+x\right)$
    g) $\displaystyle\lim_{x\to+\infty}\frac{\sqrt{x^2+1}}{x}$
    h) $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{x^2+1}}{x}$

17. Evaluate each of the following limits and explain your steps carefully:
    a) $\displaystyle\lim_{x\to 3}\frac{x^2-9}{x-3}$
    b) $\displaystyle\lim_{x\to 0}x^2\sin\left(\frac{1}{x}\right)$
    c) $\displaystyle\lim_{x\to 2}\frac{x-2}{\sqrt{2x^2+1}-3}$
    d) $\displaystyle\lim_{x\to 0}x^2\cos\left(\frac{1}{x}\right)$
    e) $\displaystyle\lim_{x\to\infty}\frac{x^2+4x+6}{4x^2+3}$
    f) $\displaystyle\lim_{x\to 0}x^2\sin\left(\frac{4}{x}\right)$

18. Evaluate each of the following limits if it exists:

a) $\displaystyle\lim_{x\to+\infty} \frac{\sqrt{9x^2+4x+5}}{x}$ b) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{9x^2+4x+5}}{x}$ c) $\displaystyle\lim_{x\to0} \frac{\sqrt{5x^2+x^4}}{x}$

d) $\displaystyle\lim_{x\to0} \frac{|\sin x|}{x}$ e) $\displaystyle\lim_{x\to0^+} \frac{\sqrt{x^2+x^3}}{x}$ f) $\displaystyle\lim_{x\to0^-} \frac{\sqrt{x^2+x^3}}{x}$

g) $\displaystyle\lim_{x\to0} \frac{\sqrt{x^2+x^3}}{x}$

19. You are given that $\sin x \to 0$ and $\cos x \to 1$ as $x \to 0$. Explain carefully how the sum, product and quotient rules of limits can be used to study the function

$$\frac{x^2+\sin x}{\cos x},$$

and calculate its limit as $x \to 0$.

20. a) Show that $\displaystyle\lim_{x\to0} \frac{\sin 4x}{\sin x\cos x} = 4$ and $\displaystyle\lim_{x\to0} \frac{\sin 2x\sin 3x}{x^2} = 6$.

b) Use the results in part (a) and the Squeezing principle, or otherwise, to show that

$$\lim_{x\to0} \left(\frac{3\sin 4x}{\sin x\cos x} - \frac{2\sin 2x\sin 3x}{x^2} \right) \sin(\mathrm{e}^{\cos x}) = 0.$$

You must explain carefully each step of your argument.