FIRST YEAR CALCULUS

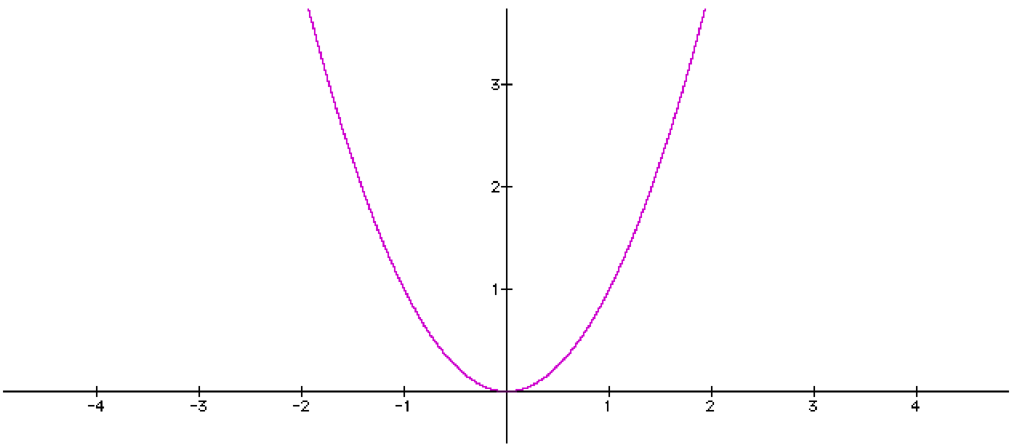
W W L CHEN

Chapter 7

CONTINUITY

7.1. Introduction

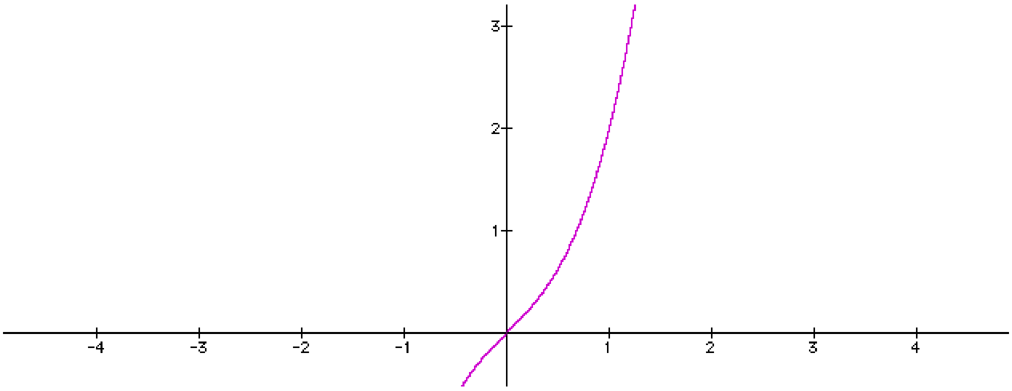
Example 7.1.1. Consider the function $f(x) = x^2$. The graph below represents this function.



It is a parabola, and we can draw this parabola without lifting our pencil from the paper.

Example 7.1.2. Consider the function $f(x) = x/|x|$, as discussed in Example 6.1.6. If we now attempt to draw the graph representing this function, then it is impossible to draw this graph without lifting our pencil from the paper. After all, there is a break, or discontinuity, at $x = 0$, where the function is not defined. Even if we were to give some value to the function at $x = 0$, then it would still be impossible to draw this graph without lifting our pencil from the paper. It is impossible to avoid the jump from the value -1 to the value 1 when we go past $x = 0$ from left to right.

EXAMPLE 7.1.3. Consider the function $f(x) = x^3 + x$. We showed in Example 6.1.9 that $f(x) \to f(1)$ as $x \to 1$. The graph represents this function.



As we approach $x = 1$ from either side, the curve goes without break towards $f(1)$. In this instance, we say that $f(x)$ is continuous at $x = 1$.

We observe from Example 7.1.3 that it is possible to formulate continuity of a function $f(x)$ at a point $x = a$ in terms of $f(a)$ and the limit of $f(x)$ at $x = a$ as follows.

DEFINITION. We say that a function $f(x)$ is continuous at $x = a$ if $f(x) \to f(a)$ as $x \to a$; in other words, if

$$\lim_{x \to a} f(x) = f(a).$$

EXAMPLE 7.1.4. The function $f(x) = x^2$ is continuous at $x = a$ for every $a \in \mathbb{R}$.

EXAMPLE 7.1.5. The function $f(x) = x/|x|$ is continuous at $x = a$ for every non-zero $a \in \mathbb{R}$. To see this, note that for every non-zero $a \in \mathbb{R}$, there is an open interval $a_1 < x < a_2$ which contains $x = a$ but not $x = 0$. The function is clearly constant in this open interval.

EXAMPLE 7.1.6. The function $f(x) = x^3 + x$ is continuous at $x = a$ for every $a \in \mathbb{R}$.

EXAMPLE 7.1.7. The function $f(x) = \sin x$ is continuous at $x = a$ for every $a \in \mathbb{R}$. To see this, note first the inequalities

$$|\sin x - \sin a| = |2 \cos \tfrac{1}{2}(x + a) \sin \tfrac{1}{2}(x - a)| \le |2 \sin \tfrac{1}{2}(x - a)| \le |x - a|$$

(here we are using the well known fact that $|\sin y| \le |y|$ for every $y \in \mathbb{R}$). It follows that given any $\epsilon > 0$, we have

$$|f(x) - f(a)| < \epsilon \quad \text{whenever } |x - a| < \min\{\epsilon, \pi\}.$$

EXAMPLE 7.1.8. It is worthwhile to mention that the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational,} \end{cases}$$

is not continuous at $x = a$ for any $a \in \mathbb{R}$. In other words, $f(x)$ is continuous nowhere. The proof is rather long and complicated. It depends on the well known fact that between any two real numbers, there are rational and irrational numbers.

Since continuity is defined in terms of limits, we have immediately the following simple consequence of Proposition 6A.

PROPOSITION 7A. *Suppose that the functions $f(x)$ and $g(x)$ are continuous at $x = a$. Then*
(a) $f(x) + g(x)$ is continuous at $x = a$;
(b) $f(x)g(x)$ is continuous at $x = a$; and
(c) if $g(a) \neq 0$, then $f(x)/g(x)$ is continuous at $x = a$.

We also have the following result concerning composition of functions. The proof is left as an exercise.

PROPOSITION 7B. *Suppose that the function $f(x)$ is continuous at $x = a$, and that the function $g(y)$ is continuous at $y = b = f(a)$. Then the composition function $(g \circ f)(x)$ is continuous at $x = a$.*

7.2. Continuity in Intervals

We have already investigated functions which are continuous at $x = a$ for a lot of values $a \in \mathbb{R}$. This observation prompts us to make definitions for stronger continuity properties. More precisely, we consider continuity in intervals, and study some of the consequences.

There is nothing special about continuity in open intervals.

Definition. Suppose that $A, B \in \mathbb{R}$ with $A < B$. We say that a function $f(x)$ is continuous in the open interval (A, B) if $f(x)$ is continuous at $x = a$ for every $a \in (A, B)$.

Remarks. (1) Suppose that a function $f(x)$ is continuous in the open interval (A, B). If we now attempt to draw the graph representing this function, but only restricted to the open interval (A, B), then we can do it without lifting our pencil from the paper.

(2) Our definition can be extended in the natural way to include open intervals of the types (A, ∞), $(-\infty, B)$ and $(-\infty, \infty)$.

Example 7.2.1. The function $f(x) = 1/x$ is continuous in the open interval $(0, 1)$. It is also continuous in the open interval $(0, \infty)$.

Example 7.2.2. The function $f(x) = x^2$ is continuous in every open interval.

Example 7.2.3. The function $f(x)$, defined by $f(0) = 1$ and $f(x) = x^{-1} \sin x$ for every $x \neq 0$, is continuous in every open interval. Note that continuity at $x = a$ for any non-zero $a \in \mathbb{R}$ can be established by combining Example 7.1.7 and Proposition 7A(c). On the other hand, continuity at $x = 0$ is a consequence of Example 6.2.3.

To formulate a suitable definition for continuity in a closed interval, we consider first an example.

Example 7.2.4. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

It is clear that this function is not continuous at $x = 0$, since

$$\lim_{x \to 0-} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to 0+} f(x) = 1.$$

However, let us investigate the behaviour of the function in the closed interval $[0, 1]$. It is clear that $f(x)$ is continuous at $x = a$ for every $a \in (0, 1)$. Furthermore, we have

$$\lim_{x \to 0+} f(x) = f(0) \qquad \text{and} \qquad \lim_{x \to 1-} f(x) = f(1).$$

Indeed, if we now attempt to draw the graph representing this function, but only restricted to the closed interval $[0, 1]$, then we can do it without lifting our pencil from the paper.

Example 7.2.4 leads us to conclude that it is not appropriate to insist on continuity of the function at the end points of the closed interval, and that a more suitable requirement is one sided continuity instead.

DEFINITION. Suppose that $A, B \in \mathbb{R}$ with $A < B$. We say that a function $f(x)$ is continuous in the closed interval $[A, B]$ if $f(x)$ is continuous in the open interval (A, B) and if

$$\lim_{x \to A+} f(x) = f(A) \qquad \text{and} \qquad \lim_{x \to B-} f(x) = f(B).$$

REMARK. It follows that for continuity of a function in a closed interval, we need right hand continuity of the function at the left hand end point of the interval, left hand continuity of the function at the right hand end point of the interval, and continuity at every point in between.

EXAMPLE 7.2.5. The function

$$f(x) = \begin{cases} x + 2 & \text{if } x \geq 1, \\ x + 1 & \text{if } x < 1, \end{cases}$$

is continuous in the closed interval $[1, 2]$, but not continuous in the closed interval $[0, 1]$.

7.3. Continuity in Closed Intervals

Let us draw the graph of the function $f(x) = 1/x$ in the open interval $(0, 1)$. Recall that $f(x)$ is continuous in $(0, 1)$. As $x \to 0+$, we clearly have $f(x) \to +\infty$. It follows that $f(x)$ cannot have a finite maximum value in the open interval $(0, 1)$. For every $M \in \mathbb{R}$, we can always choose x small enough so that $f(x) = 1/x > M$.

Such a phenomenon cannot happen for a function continuous in a closed interval. Suppose that a function $f(x)$ is continuous in the closed interval $[A, B]$. Imagine that we are drawing the graph of $f(x)$ in $[A, B]$. Let us start at the point $(A, f(A))$. We hope to reach the point $(B, f(B))$ without lifting our pencil from the paper. We would not succeed if the graph were to go off to infinity somewhere in between.

This observation is summarized by the following result which we shall prove later in this section.

PROPOSITION 7C. (MAX-MIN THEOREM) *Suppose that a function $f(x)$ is continuous in the closed interval $[A, B]$, where $A, B \in \mathbb{R}$ with $A < B$. Then there exist real numbers $x_1, x_2 \in [A, B]$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for every $x \in [A, B]$. In other words, the function $f(x)$ attains a maximum value and a minumum value in the closed interval $[A, B]$.*

EXAMPLE 7.3.1. Consider the function $f(x) = \cos x$ in the closed interval $[-1, \pi/3]$. If we draw the graph of $f(x)$ in the closed interval $[-1, \pi/3]$, then it is not difficult to see that $f(\pi/3) \leq f(x) \leq f(0)$ for every $x \in [-1, \pi/3]$.

EXAMPLE 7.3.2. Consider the function $f(x) = \cos x$ in the closed interval $[-20\pi, 20\pi]$. It is not difficult to see that $f(7\pi) \leq f(x) \leq f(-16\pi)$ for every $x \in [-20\pi, 20\pi]$. In fact, it can be checked that $f(x)$ attains its maximum value at 21 different values of $x \in [-20\pi, 20\pi]$ and attains its minimum value at 20 different values of $x \in [-20\pi, 20\pi]$.

REMARK. Our last example shows that the points $x_1, x_2 \in [A, B]$ in Proposition 7C may not be unique.

Suppose that a function $f(x)$ is continuous in the closed interval $[A, B]$, and that we have drawn the graph of $f(x)$ in $[A, B]$. Suppose further that we have located real numbers $x_1, x_2 \in [A, B]$ such that $f(x_1) \le f(x) \le f(x_2)$ for every $x \in [A, B]$, so that $f(x_1)$ is the minimum value of $f(x)$ in $[A, B]$ and $f(x_2)$ is the maximum value of $f(x)$ in $[A, B]$. Suppose next that $y \in \mathbb{R}$ satisfies $f(x_1) < y < f(x_2)$; in other words, y is any real number between the maximum value and the minimum value of $f(x)$ in $[A, B]$. Let us draw a horizontal line at height y, so that the two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ are on opposite sides of this line. If we start at the point $(x_1, f(x_1))$ and follow the graph of $f(x)$ towards the point $(x_2, f(x_2))$, then we clearly must meet this horizontal line somewhere along the way. Furthermore, if this meeting point is (x_0, y), then clearly $y = f(x_0)$.

This is an illustration of the following important result which we shall establish shortly.

PROPOSITION 7D. (INTERMEDIATE VALUE THEOREM) *Suppose that a function $f(x)$ is continuous in the closed interval $[A, B]$, where $A, B \in \mathbb{R}$ with $A < B$. Suppose further that the real numbers $x_1, x_2 \in [A, B]$ satisfy $f(x_1) \le f(x) \le f(x_2)$ for every $x \in [A, B]$. Then for every real number $y \in \mathbb{R}$ satisfying $f(x_1) \le y \le f(x_2)$, there exists a real number $x_0 \in [A, B]$ such that $f(x_0) = y$.*

EXAMPLE 7.3.3. Consider the function $f(x) = x + 3x^2 \sin x$. It is not difficult to see that $f(x)$ is continuous at every $x \in \mathbb{R}$, and so continuous in every closed interval. Note that $f(-\pi) < 0$ and $f(-3\pi/2) > 0$. Now consider the function $f(x)$ in the closed interval $[-3\pi/2, -\pi]$. By the Intermediate value theorem, we know that there exists $x_0 \in [-3\pi/2, -\pi]$ such that $f(x_0) = 0$. In other words, we have shown that there is a root of the equation $x + 3x^2 \sin x = 0$ in the interval $[-3\pi/2, -\pi]$.

EXAMPLE 7.3.4. Consider the function $f(x) = x^3 - 3x - 1$. Clearly $f(-1) = 1 > 0$ and $f(0) = -1 < 0$. It is easy to check that $f(x)$ is continuous in the closed interval $[-1, 0]$. By the Intermediate value theorem, we know that there exists $x_0 \in [-1, 0]$ such that $f(x_0) = 0$. In other words, we have shown that there is a root of the equation $x^3 - 3x - 1 = 0$ in the interval $[-1, 0]$.

To establish Propositions 7C and 7D, it is convenient to make the following definition.

DEFINITION. Suppose that a function $f(x)$ is defined on an interval $I \subseteq \mathbb{R}$. We say that $f(x)$ is bounded above on I if there exists a real number $K \in \mathbb{R}$ such that $f(x) \le K$ for every $x \in I$, and that $f(x)$ is bounded below on I if there exists a real number $k \in \mathbb{R}$ such that $f(x) \ge k$ for every $x \in I$. Furthermore, we say that $f(x)$ is bounded on I if it is bounded above and bounded below on I.

We shall first of all establish the following result.

PROPOSITION 7E. *Suppose that a function $f(x)$ is continuous in the closed interval $[A, B]$, where $A, B \in \mathbb{R}$ with $A < B$. Then $f(x)$ is bounded on $[A, B]$.*

PROOF. Consider the set

$$S = \{C \in [A, B] : f(x) \text{ is bounded on } [A, C]\}.$$

Then S is non-empty, since clearly $A \in S$. On the other hand, S is bounded above by B. It follows from the Completeness axiom that S has a supremum. Let $\xi = \sup S$. Clearly $\xi \le B$. We shall first of all show that $\xi = B$. Suppose not. Then either $\xi = A$ or $A < \xi < B$. We shall consider the second possibility – the argument for the first case needs only minor modifications. Since $f(x)$ is continuous at $x = \xi$, there exists $\delta > 0$ such that $\xi - \delta \ge A$ and

$$|f(x) - f(\xi)| < 1 \quad \text{whenever } |x - \xi| < \delta,$$

so that

$$|f(x)| < |f(\xi)| + 1 \quad \text{whenever } \xi - \delta < x < \xi + \delta.$$

Clearly $\xi - \delta \in S$, so that $f(x)$ is bounded on $[A, \xi - \delta]$. If $|f(x)| \leq M$ for every $x \in [A, \xi - \delta]$, then

$$|f(x)| \leq \max\{M, |f(\xi)| + 1\} \quad \text{whenever } x \in [A, \xi + \tfrac{1}{2}\delta],$$

so that $\xi + \frac{1}{2}\delta \in S$, contradicting the assumption that $\xi = \sup S$.

Next, we know that $f(x)$ is left continuous at $x = B$, so there exists $\delta > 0$ such that $B - \delta > A$ and

$$|f(x) - f(B)| < 1 \quad \text{whenever } B - \delta < x \leq B,$$

so that

$$|f(x)| < |f(B)| + 1 \quad \text{whenever } B - \delta < x \leq B.$$

Clearly $B - \delta \in S$, so that $f(x)$ is bounded on $[A, B - \delta]$. If $|f(x)| \leq K$ for every $x \in [A, B - \delta]$, then

$$|f(x)| \leq \max\{K, |f(B)| + 1\} \quad \text{whenever } x \in [A, B],$$

and this completes the proof. ◯

PROOF OF PROPOSITION 7C. We shall only establish the existence of the real number $x_2 \in [A, B]$, as the existence of the real number $x_1 \in [A, B]$ can be established by repeating the argument here on the function $-f(x)$. Note first of all that it follows from Proposition 7E that the set

$$S = \{f(x) : x \in [A, B]\}$$

is bounded above. Let $M = \sup S$. Then $f(x) \leq M$ for every $x \in [A, B]$. Suppose on the contrary that there does not exist $x_2 \in [A, B]$ such that $f(x) \leq f(x_2)$ for every $x \in [A, B]$. Then $f(x) < M$ for every $x \in [A, B]$, and so it follows from Proposition 7A that the function

$$g(x) = \frac{1}{M - f(x)}$$

is continuous in the closed interval $[A, B]$, and is therefore bounded above on $[A, B]$ as a consequence of Proposition 7E. Suppose that $g(x) \leq K$ for every $x \in [A, B]$. Since $g(x) > 0$ for every $x \in [A, B]$, we must have $K > 0$. But then the inequality $g(x) \leq K$ gives the inequality

$$f(x) \leq M - \frac{1}{K},$$

contradicting the assumption that $M = \sup S$. ◯

PROOF OF PROPOSITION 7D. We may clearly suppose that $f(x_1) < y < f(x_2)$. By considering the function $-f(x)$ if necessary, we may further assume, without loss of generality, that $x_1 < x_2$. The idea of the proof is then to follow the graph of the function $f(x)$ from the point $(x_1, f(x_1))$ to the point $(x_2, f(x_2))$. This clearly touches the horizontal line at height y at least once; the reader is advised to draw a picture. Our technique is then to trap the last occasion when this happens. Accordingly, we consider the set

$$T = \{x \in [x_1, x_2] : f(x) \leq y\}.$$

This set is clearly bounded above. Let $x_0 = \sup T$. We shall show that $f(x_0) = y$. Suppose on the contrary that $f(x_0) \neq y$. Then exactly one of the following two cases applies:

(a) We have $f(x_0) > y$. In this case, let $\epsilon = f(x_0) - y > 0$. Since $f(x)$ is continuous at $x = x_0$, it follows that there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. This implies that

$f(x) > y$ for every real number $x \in (x_0 - \delta, x_0 + \delta)$, so that $x_0 - \delta$ is an upper bound of T, contradicting the assumption that $x_0 = \sup T$.

(b) We have $f(x_0) < y$. In this case, let $\epsilon = y - f(x_0) > 0$. Since $f(x)$ is continuous at $x = x_0$, it follows that there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. This implies that $f(x) < y$ for every real number $x \in (x_0 - \delta, x_0 + \delta)$, so that x_0 cannot be an upper bound of T, again contradicting the assumption that $x_0 = \sup T$. ◯

7.4. An Application to Numerical Mathematics

In this section, we outline a very simple technique for finding approximations to solutions of equations. This technique is based on repeated application of the Intermediate value theorem. In fact, in our previous two examples, we have already taken the first step.

The technique is sometimes known as the Bisection technique, and is based on the simple observation that a non-zero real number must be positive or negative, but not both.

BISECTION TECHNIQUE. *Suppose that a function $f(x)$ is continuous in the closed interval $[A, B]$, where $A, B \in \mathbb{R}$ with $A < B$. Suppose further that $f(A)f(B) < 0$. Clearly $f(A)$ and $f(B)$ are non-zero and have different signs. By the Intermediate value theorem, we know that there is a solution of the equation $f(x) = 0$ in the interval (A, B). We calculate $f(C)$, where $C = (A + B)/2$ is the midpoint of the interval $[A, B]$. Exactly one of the following holds:*
(1) If $f(C) = 0$, then we have found a solution to the equation $f(x) = 0$, and the process ends.
(2) If $f(A)f(C) < 0$, then we repeat all the steps above by considering the function $f(x)$ in the closed interval $[A, C]$.
(3) If $f(B)f(C) < 0$, then we repeat all the steps above by considering the function $f(x)$ in the closed interval $[C, B]$.

Remark. Note that if the process does not end, then on each application, we have halved the length of the interval under discussion. It follows that after k applications, the interval is only 2^{-k} times the length of the original interval. Hence this very simple technique is rather efficient.

Example 7.4.1. Consider again the function $f(x) = x^3 - 3x - 1$. Try to represent the following information in a picture in order to understand the technique.

- We have $f(-1) > 0$ and $f(0) < 0$. By the Intermediate value theorem, we know that there is a solution of the equation $f(x) = 0$ in the interval $(-1, 0)$. Now $f(-0.5) > 0$, so we repeat the process by considering the function $f(x)$ in the closed interval $[-0.5, 0]$.
- We have $f(-0.5) > 0$ and $f(0) < 0$. By the Intermediate value theorem, we know that there is a solution of the equation $f(x) = 0$ in the interval $(-0.5, 0)$. Now $f(-0.25) < 0$, so we repeat the process by considering the function $f(x)$ in the closed interval $[-0.5, -0.25]$.
- We have $f(-0.5) > 0$ and $f(-0.25) < 0$. By the Intermediate value theorem, we know that there is a solution of the equation $f(x) = 0$ in the interval $(-0.5, -0.25)$. Now $f(-0.375) > 0$, so we repeat the process by considering the function $f(x)$ in the closed interval $[-0.375, -0.25]$.
- We have $f(-0.375) > 0$ and $f(-0.25) < 0$. By the Intermediate value theorem, we know that there is a solution of the equation $f(x) = 0$ in the interval $(-0.375, -0.25)$. Now $f(-0.3125) < 0$, so we repeat the process by considering the function $f(x)$ in the closed interval $[-0.375, -0.3125]$.
- We have $f(-0.375) > 0$ and $f(-0.3125) < 0$. By the Intermediate value theorem, we know that there is a solution of the equation $f(x) = 0$ in the interval $(-0.375, -0.3125)$. Now $f(-0.34375) < 0$, so we repeat the process by considering the function $f(x)$ in the closed interval $[-0.375, -0.34375]$.
- We have $f(-0.375) > 0$ and $f(-0.34375) < 0$. By the Intermediate value theorem, we know that there is a solution of the equation $f(x) = 0$ in the interval $(-0.375, -0.34375)$. Of course, a few more applications will lead to yet smaller intervals, and so better approximations.

7.5. An Application to Inequalities

In this section, we outline a justification for a simple technique which enables us to determine those values of x for which a given quantity $p(x)$ is positive (or negative) when it is possible to determine all the solutions of the equation $p(x) = 0$ and all the discontinuities of $p(x)$. We illustrate this technique with an example.

EXAMPLE 7.5.1. We wish to determine precisely those values of $x \in \mathbb{R}$ for which the inequality

$$\frac{x^2 + 7x + 2}{x - 3} > 1$$

holds. This inequality can be rewritten in the equivalent form $p(x) > 0$, where the function

$$p(x) = \frac{x^2 + 7x + 2}{x - 3} - 1$$

has a discontinuity at the point $x = 3$ and is continuous at every other point. Let us find the roots of the equation $p(x) = 0$. It is easy to see that they are precisely the roots of the polynomial equation $x^2+6x+5 = 0$, and so the roots are $x = -1$ and $x = -5$. We now have to consider the intervals $(-\infty, -5)$, $(-5, -1)$, $(-1, 3)$ and $(3, \infty)$, and proceed to choose representatives -6, -2, 0 and 4 respectively, say, from these intervals and study the sign of each of $p(-6)$, $p(-2)$, $p(0)$ and $p(4)$. It is easy to see that $p(-6) < 0$, $p(-2) > 0$, $p(0) < 0$ and $p(4) > 0$, so we conclude that

$$p(x) \begin{cases} < 0 & \text{if } x < -5 \text{ or } -1 < x < 3; \\ > 0 & \text{if } -5 < x < -1 \text{ or } x > 3. \end{cases}$$

Hence the given inequality holds precisely when $-5 < x < -1$ or $x > 3$.

It appears that we have made a conclusion about the sign of $p(x)$ in an interval by simply checking the sign of $p(x)$ at one point within the interval. That we can do this is a consequence of the Intermediate value theorem. Suppose that the function $p(x)$ is non-zero and has no discontinuity in the interval (A, B). Suppose on the contrary that $x_1, x_2 \in (A, B)$ satisfy $p(x_1) < 0$ and $p(x_2) > 0$. Applying the Intermediate value theorem on the closed interval with endpoints x_1 and x_2, we conclude that there must be some x_0 between x_1 and x_2 such that $p(x_0) = 0$, contradicting the assumption that $p(x)$ is non-zero in the interval (A, B).

Problems for Chapter 7

1. Prove that each of the following functions is continuous at $x = 0$:
   a) $f(x) = [x^2]$
   b) $g(x) = x \sin(1/x)$ when $x \neq 0$, and $g(0) = 0$

2. Find a and b so that the function

$$f(x) = \begin{cases} -x^2 + a & \text{if } x \le 0, \\ x \sin \dfrac{1}{x} + 1 & \text{if } 0 < x \le \dfrac{1}{\pi}, \\ bx^3 + 2 & \text{if } \dfrac{1}{\pi} < x, \end{cases}$$

   is continuous everywhere.

3. Find a and b so that the function

$$f(x) = \begin{cases} -x^3 + 1 & \text{if } x < 0, \\ ax + b & \text{if } 0 \le x \le 1, \\ \sqrt{x} + 2 & \text{if } x > 1, \end{cases}$$

   is continuous everywhere.

4. Find a and b so that the function

$$f(x) = \begin{cases} -x^3 + a & \text{if } x < 0, \\ x + b & \text{if } 0 \le x \le 1, \\ \sqrt{x} + 4 & \text{if } x > 1, \end{cases}$$

   is continuous everywhere.

5. a) Find the range of the function $f(x) = [x] - x$ in the interval $[0, 1]$.
   b) Do there exist $x_1, x_2 \in [0, 1]$ such that $f(x_1) \le f(x) \le f(x_2)$ for every $x \in [0, 1]$?
   c) Comment on the results.

6. Suppose that the function $f(x)$ is continuous in the closed interval $[0, 1]$, and that $0 \le f(x) \le 1$ for every $x \in [0, 1]$. Show that there exists $c \in [0, 1]$ such that $f(c) = c$.

7. Show that at any given time there are always antipodal points on the earth's equator with the same temperature.
   [Hint: Suppose that $f(x)$ is a continuous function in the closed interval $[0, 1]$ with $f(0) = f(1)$. Show that there exists $c \in [0, 1]$ such that $f(c) = f(c + \frac{1}{2})$.]

8. Consider the function $f(x) = x^2 - 2x \sin x - 1$, which is continuous everywhere in $\mathbb{R}$.
   a) Evaluate $f(0)$.
   b) Find some real number $A < 0$ such that $f(A) > 0$. Use the Intermediate value theorem to show that there exists a real number $\alpha < 0$ such that $f(\alpha) = 0$.
   c) Find some real number $B > 0$ such that $f(B) > 0$. Use the Intermediate value theorem to show that there exists a real number $\beta > 0$ such that $f(\beta) = 0$.

9. Given $f(x) = x^3 + 5x^2 - 4x - 1$. Find the values $f(0)$ and $f(1)$. Show that the equation $f(x) = 0$ has at least one root between 0 and 1.

10. Prove that the equation $\mathrm{e}^x = 2 - x$ has at least one real root.

11. Suppose that $a, b, c, d \in \mathbb{R}$ and $a > 0$. Use the intermediate value theorem to show that the equation $ax^3 + bx^2 + cx + d = 0$ has at least one real root.

12. Suppose that $f(x)$ is a polynomial of even degree. Prove that $f(x) \to +\infty$ as $x \to \infty$ or $f(x) \to -\infty$ as $x \to \infty$. Deduce that $f(x)$ has either a least value or a greatest value, but not both.
[HINT: Consider $f(x)$ in an interval $[-A, A]$, where A is so large that $|f(x)| > |f(0)|$ if $|x| > A$.]

13. Suppose that $f(x)$ is a polynomial of odd degree. Show that for every $y \in \mathbb{R}$, the equation $f(x) = y$ has a solution with $x \in \mathbb{R}$.
[HINT: Find a real number A so large that y lies between $f(A)$ and $f(-A)$.]