FIRST YEAR CALCULUS

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Chapter 8

DIFFERENTIATION

8.1. Elementary Results on Derivatives

Recall that if a function $f(x)$ is differentiable at $x = a$, then

$$\frac{f(x) - f(a)}{x - a} \to f'(a)$$

as $x \to a$, where $f'(a)$ is the derivative. On the other hand, clearly the function $x - a \to 0$ as $x \to a$. By the product rule of limits, we have

$$f(x) - f(a) = \left(\frac{f(x) - f(a)}{x - a} \right) (x - a) \to 0$$

as $x \to a$. It follows that $f(x) \to f(a)$ as $x \to a$. We have proved the following result.

PROPOSITION 8A. *Suppose that a function $f(x)$ is differentiable at $x = a$. Then $f(x)$ is continuous at $x = a$.*

As is in the case of limits and continuity, we have the sum, product and quotient rules for derivatives. The following result is stated as Proposition 3B earlier.

PROPOSITION 8B. *Suppose that the functions $f(x)$ and $g(x)$ are differentiable at $x = a$. Then*
(a) $f(x) + g(x)$ is differentiable at $x = a$, with $(f + g)'(a) = f'(a) + g'(a)$;
(b) $f(x)g(x)$ is differentiable at $x = a$, with $(fg)'(a) = f(a)g'(a) + f'(a)g(a)$; and
(c) if $g(a) \neq 0$, then $f(x)/g(x)$ is differentiable at $x = a$, with $\left(\frac{f}{g} \right)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{g^2(a)}$.

Proof. (a) Note that

$$\frac{(f(x)+g(x))-(f(a)+g(a))}{x-a}=\frac{f(x)-f(a)}{x-a}+\frac{g(x)-g(a)}{x-a}.$$

It follows from Proposition 6A that

$$\lim_{x\to a}\frac{(f(x)+g(x))-(f(a)+g(a))}{x-a}=f'(a)+g'(a).$$

(b) Note that

$$\begin{aligned}\frac{f(x)g(x)-f(a)g(a)}{x-a}&=\frac{f(x)g(x)-f(x)g(a)+f(x)g(a)-f(a)g(a)}{x-a}\\&=f(x)\frac{g(x)-g(a)}{x-a}+g(a)\frac{f(x)-f(a)}{x-a}.\end{aligned}$$

In view of Proposition 8A, we clearly have $f(x)\to f(a)$ as $x\to a$. It follows from Proposition 6A that

$$\lim_{x\to a}\frac{f(x)g(x)-f(a)g(a)}{x-a}=f(a)g'(a)+g(a)f'(a).$$

(c) We shall first show that $1/g(x)$ is differentiable at $x=a$. Note that

$$\frac{(1/g(x))-(1/g(a))}{x-a}=-\frac{g(x)-g(a)}{x-a}\frac{1}{g(x)}\frac{1}{g(a)}.$$

In view of Proposition 8A, we clearly have $g(x)\to g(a)$ as $x\to a$. It follows from Proposition 6A that

$$\lim_{x\to a}\frac{(1/g(x))-(1/g(a))}{x-a}=-\frac{g'(a)}{g^2(a)}.$$

We now apply part (b) to $f(x)$ and $1/g(x)$ to get the desired result. ○

As in Chapter 3, we shall from now on slightly abuse our notation, and simply refer to $f'(x)$ as the derivative of the function $f(x)$. We shall further write

$$y=f(x)\qquad\text{and}\qquad\frac{\mathrm{d}y}{\mathrm{d}x}=f'(x).$$

The following result on differentiation of composite functions, known as the Chain rule for differentiation, is stated as Proposition 3C earlier.

PROPOSITION 8C. *Suppose that y is a differentiable function of u, and that u is a differentiable function of x. Then y is a differentiable function of x. Furthermore, we have*

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Proof. Write $y=g(u)$, $u=f(x)$ and $b=f(a)$. Then $y=(g\circ f)(x)$. Note that

$$\frac{(g\circ f)(x)-(g\circ f)(a)}{x-a}=\frac{(g\circ f)(x)-(g\circ f)(a)}{f(x)-f(a)}\frac{f(x)-f(a)}{x-a}=\frac{g(u)-g(b)}{u-b}\frac{f(x)-f(a)}{x-a}.$$

Here it is tempting to deduce the conclusion immediately. However, it is possible that $u - b = 0$. To overcome this difficulty, let us introduce the function

$$G(u) = \begin{cases} \dfrac{g(u) - g(b)}{u - b} & \text{if } u \neq b, \\ g'(b) & \text{if } u = b. \end{cases}$$

Since $g(u)$ is differentiable at $u = b$, we have $G(u) \to g'(b)$ as $u \to b$. Furthermore, since $G(b) = g'(b)$, it follows that $G(u)$ is continuous at $u = b$. On the other hand, as $x \to a$, we have $u \to b$, so that $G(u) \to g'(b)$. Hence

$$G(u) \to g'(b) \quad \text{as } x \to a.$$

Suppose now that $u \neq b$. Then we clearly have

$$\frac{(g \circ f)(x) - (g \circ f)(a)}{x - a} = G(u) \frac{f(x) - f(a)}{x - a}.$$

Note that this also holds when $u = b$, since both sides are equal to 0. It now follows that

$$\lim_{x \to a} \frac{(g \circ f)(x) - (g \circ f)(a)}{x - a} = g'(b) f'(a) = g'(f(a)) f'(a)$$

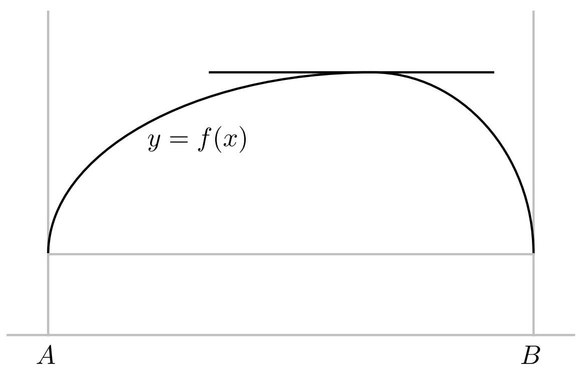
as required. ○

8.2. Two Important Results on Derivatives

As we have already shown, the derivative $f'(x)$ is very useful in studying properties of a function $f(x)$. In this section, we indicate two results which summarize, with more rigour, this rather precise link.

Try to picture the following situation. Imagine that a function $f(x)$ is continuous in the closed interval $[A, B]$, and that $f(A) = f(B)$, so that the line joining the points $(A, f(A))$ and $(B, f(B))$ is horizontal. Suppose further that $f'(x)$ exists for every $x \in (A, B)$; in other words, there is a tangent to the curve everywhere between A and B. Let us concentrate on how the tangent behaves as we move from A to B. It is not too difficult to imagine that the tangent may be horizontal at some point. After all, what goes up must come down, and what happens between going up and coming down?

PROPOSITION 8D. (ROLLE'S THEOREM) *Suppose that a function $f(x)$ is continuous in the closed interval $[A, B]$, where $A, B \in \mathbb{R}$ with $A < B$. Suppose further that $f'(a)$ exists for every $a \in (A, B)$. If $f(A) = f(B)$, then there exists $c \in (A, B)$ such that $f'(c) = 0$.*



EXAMPLE 8.2.1. We can prove that between any two real roots of $\sin x = 0$ must lie a real root of $\cos x = 0$. To do this, let $f(x) = \sin x$, and let $A < B$ be any two real roots of $\sin x = 0$. Clearly $f(A) = f(B)$. Furthermore, all the other hypotheses of Rolle's theorem are satisfied. It follows that there exists $c \in (A, B)$ such that $f'(c) = 0$. Note, however, that $f'(x) = \cos x$.

EXAMPLE 8.2.2. Consider the polynomial $f(x) = x^3 + 3x^2 + 6x + 1$. We can prove that the polynomial equation $f(x) = 0$ has exactly one real root. Note that $f(-1) < 0$ and $f(1) > 0$. Applying the Intermediate value theorem to $f(x)$ in the closed interval $[-1, 1]$, we know that there exists $x_0 \in (-1, 1)$ such that $f(x_0) = 0$. It follows that the equation $f(x) = 0$ has at least one real root. Suppose that there are more than one real root. Let $A < B$ be two such roots. Then clearly $f(A) = f(B)$. Applying Rolle's theorem with $f(x) = x^3 + 3x^2 + 6x + 1$ in the interval $[A, B]$, we conclude that there exists $c \in (A, B)$ such that $f'(c) = 0$. Note, however, that $f'(x) = 3x^2 + 6x + 6 = 3(x^2 + 2x + 1 + 1) = 3(x + 1)^2 + 3 \neq 0$ for any $x \in \mathbb{R}$.

PROOF OF PROPOSITION 8D. Since $f(x)$ is continuous in the closed interval $[A, B]$, it follows from Proposition 7C that there exist $x_1, x_2 \in [A, B]$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for every $x \in [A, B]$.

Case 1. Suppose that both x_1 and x_2 are endpoints of the interval $[A, B]$. Since $f(A) = f(B)$, it follows that $f(x)$ is constant in the interval $[A, B]$, so that $f'(c) = 0$ for every $c \in (A, B)$.

Case 2. Suppose that $x_1 \in (A, B)$. Then $f(x)$ has a local minimum at $x = x_1$. We claim that $f'(x_1) = 0$. Suppose on the contrary that $f'(x_1) \neq 0$. Without loss of generality, assume that

$$f'(x_1) = \lim_{x \to x_1} \frac{f(x) - f(x_1)}{x - x_1} > 0.$$

Then there exists $\delta > 0$ such that

$$\left| \frac{f(x) - f(x_1)}{x - x_1} - f'(x_1) \right| < \frac{1}{2} |f'(x_1)| \quad \text{whenever } 0 < |x - x_1| < \delta,$$

so that

$$\frac{f(x) - f(x_1)}{x - x_1} > 0 \quad \text{whenever } 0 < |x - x_1| < \delta.$$

It follows that $f(x) - f(x_1) < 0$ if $x_1 - \delta < x < x_1$, contradicting that $f(x)$ has a local minimum at $x = x_1$.

Case 3. Suppose that $x_2 \in (A, B)$. Then $f(x)$ has a local maximum at $x = x_2$. A similar argument as in Case 2 gives $f'(x_2) = 0$. ◯

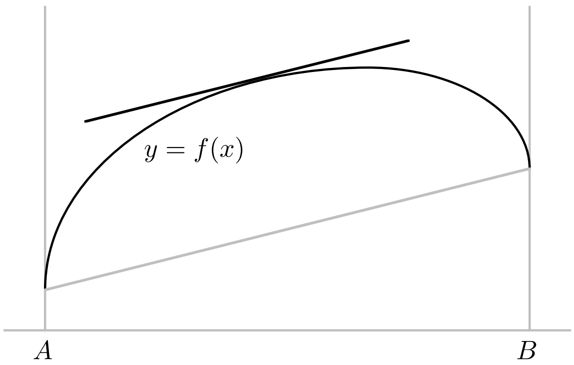
The hypotheses of Rolle's theorem are rather restrictive, since we require that the function has equal values at the two end points of the interval in question. If we relax this restriction, then our conclusion will be naturally weaker. However, this new version is much more useful, and is stated earlier as Proposition 3F.

PROPOSITION 8E. (MEAN VALUE THEOREM) *Suppose that a function $f(x)$ is continuous in the closed interval $[A, B]$, where $A, B \in \mathbb{R}$ with $A < B$. Suppose further that $f'(a)$ exists for every $a \in (A, B)$. Then there exists $c \in (A, B)$ such that $f(B) - f(A) = f'(c)(B - A)$.*

To understand the Mean value theorem, it is easiest to rewrite the conclusion as

$$\frac{f(B) - f(A)}{B - A} = f'(c).$$

The left hand side represents the slope of the line joining the points $(A, f(A))$ and $(B, f(B))$. It follows that the theorem merely says that the tangent to the curve is sometimes parallel to this line.



It is therefore clear that Rolle's theorem is a special case of the Mean value theorem. We next show that the Mean value theorem can be deduced fairly easily from Rolle's theorem.

Proof of Proposition 8E. Consider the function

$$g(x) = f(x) - \frac{f(B) - f(A)}{B - A}(x - A).$$

Then clearly $g(x)$ is continuous in the closed interval $[A, B]$, $g'(a)$ exists for every $a \in (A, B)$ and $g(A) = g(B)$. It follows from Rolle's theorem that there exists $c \in (A, B)$ such that $g'(c) = 0$. Note now that

$$g'(c) = f'(c) - \frac{f(B) - f(A)}{B - A}.$$

This completes the proof. ○

8.3. Consequences of the Mean Value Theorem

The Mean value theorem allows us to draw conclusions about the behaviour of a function through knowledge of its derivative. An example of this is given by the following result stated and established earlier as Proposition 3G.

PROPOSITION 8F. *Suppose that a function $f(x)$ is continuous in the closed interval $[A, B]$, where $A, B \in \mathbb{R}$ with $A < B$. Suppose further that $f'(a)$ exists for every $a \in (A, B)$.*
(a) If $f'(a) = 0$ for every $a \in (A, B)$, then $f(x)$ is constant in $[A, B]$.
(b) If $f'(a) > 0$ for every $a \in (A, B)$, then $f(x)$ is strictly increasing in $[A, B]$.
(c) If $f'(a) < 0$ for every $a \in (A, B)$, then $f(x)$ is strictly decreasing in $[A, B]$.

An immediate consequence is the following result stated earlier as Proposition 3D.

PROPOSITION 8G. *Suppose that I is an open interval containing a. Suppose further that a function $f(x)$ is continuous in I, and differentiable at every $x \in I$, except possibly at $x = a$.*
(a) If $f'(x) > 0$ for every $x < a$ in I and $f'(x) < 0$ for every $x > a$ in I, then the function $f(x)$ has a local maximum at $x = a$.
(b) If $f'(x) < 0$ for every $x < a$ in I and $f'(x) > 0$ for every $x > a$ in I, then the function $f(x)$ has a local minimum at $x = a$.

PROOF. (a) It follows from Proposition 8F that $f(x)$ is strictly increasing to the left of $x = a$ and strictly decreasing to the right of $x = a$, so that $f(x)$ clearly has a local maximum at $x = a$.

(b) It follows from Proposition 8F that $f(x)$ is strictly decreasing to the left of $x = a$ and strictly increasing to the right of $x = a$, so that $f(x)$ clearly has a local minimum at $x = a$. ◯

We can also establish the following result concerning second derivatives stated as Proposition 3E earlier.

PROPOSITION 8H. *Suppose that I is an open interval containing a real number a. Suppose further that the function $f(x)$ is differentiable at every $x \in I$, and that $f'(a) = 0$.*
(a) If $f''(a) < 0$, then the function $f(x)$ has a local maximum at $x = a$.
(b) If $f''(a) > 0$, then the function $f(x)$ has a local minimum at $x = a$.

PROOF. We shall only prove (a), as the proof for (b) is similar. Since

$$f''(a) = \lim_{x \to a} \frac{f'(x) - f'(a)}{x - a} < 0,$$

it follows that there exists $\delta > 0$ such that

$$\left| \frac{f'(x) - f'(a)}{x - a} - f''(a) \right| < \frac{1}{2}|f''(a)| \quad \text{whenever } 0 < |x - a| < \delta,$$

so that

$$\frac{f'(x) - f'(a)}{x - a} < 0 \quad \text{whenever } 0 < |x - a| < \delta.$$

Now let $I = (a - \delta, a + \delta)$. Then it is easy to see that $f'(x) > 0$ for every $x < a$ in I and $f'(x) < 0$ for every $x > a$ in I. It now follows from Proposition 8G that $f(x)$ has a local maximum at $x = a$. ◯

Problems for Chapter 8

1. Suppose that the function $f(x)$ satisfies $f(0) = 0$, $f'(0) = 0$ and $f''(0) > 0$.
   a) Explain why there exists $\delta > 0$ such that $\dfrac{f'(x) - f'(0)}{x - 0} > 0$ for every non-zero $x \in (-\delta, \delta)$.
   b) Deduce that $f'(x) > 0$ for every $x \in (0, \delta)$, and that $f'(x) < 0$ for every $x \in (-\delta, 0)$.
   c) Use Rolle's theorem to show that $f(x) \neq 0$ for every non-zero $x \in (-\delta, \delta)$.
   d) Use the Mean value theorem to show that $f(x) > 0$ for every non-zero $x \in (-\delta, \delta)$.

2. Consider the function $f(x) = x^{2/3}$ in the closed interval $[-1, 1]$.
   a) Show that $f(-1) = f(1)$.
   b) Show that there is no number $c \in (-1, 1)$ such that $f'(c) = 0$.
   c) Show that $f(x)$ is not differentiable at $x = 0$.
   d) Explain why the conclusion of Rolle's theorem does not hold.

3. Explain why $x = 1$ is the only real solution of the equation $x^3 - 3x^2 + 9x - 7 = 0$.

4. Let
$$f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$
   a) Show that $f(x)$ is continuous at $x = 0$.
   b) Find the derivative of $f(x)$ when $x \neq 0$.
   c) Show that $f(x)$ is not differentiable at $x = 0$.

5. Use the relevant theorems to prove that the equation $\mathrm{e}^x = 3 - x$ has exactly one real solution.

6. Show that the equation $3x - 2 + \cos \dfrac{\pi x}{2} = 0$ has exactly one real root.

7. Suppose that the functions $f(x)$ and $g(x)$ are continuous in the closed interval $[A, B]$ and differentiable in the open interval (A, B). Suppose further that $g'(x) \neq 0$ for every $x \in (A, B)$.
   a) By considering a function $\phi(x) = f(x) - kg(x)$, where the constant k is suitably chosen, and using Rolle's theorem, show that there exists $c \in (A, B)$ such that
$$\frac{f'(c)}{g'(c)} = \frac{f(B) - f(A)}{g(B) - g(A)}.$$
   [Remark: This is the Cauchy mean value theorem.]
   b) Suppose further that $f(A) = g(A) = 0$. Deduce that
$$\lim_{x \to A+} \frac{f(x)}{g(x)} = \lim_{x \to A+} \frac{f'(x)}{g'(x)},$$
   provided that the second limit exists.
   [Remark: This is one version of l'Hopital's rule.]

8. Use the mean value theorem to show that if the derivative $f'(x) < 0$ for all x in the open interval (A, B), then the function $f(x)$ is decreasing on (A, B).

9. Use the Mean Value Theorem to prove the inequality $|\sin A - \sin B| \leq |A - B|$ for all real numbers A and B.

10. Let $f(x) = \tan x - x$. Find $f(0)$ and use the derivative $f'(x)$ to prove that $\tan x > x$ for every x satisfying $0 < x < \pi/2$.

11. Let
$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$
 a) Prove that $f'(x)$ exists for every real number x.
 b) Find $f'(0)$.
 c) Find $f'(x)$ when $x \neq 0$.
 d) Prove that $f'(x)$ is not continuous at $x = 0$.

12. Construct a function $g(x)$ for which $g'(0) > 0$, but there is no interval $(-A, A)$ in which $g(x)$ is an increasing function.
[HINT: Try $g(x) = f(x) + kx$, where k is a suitable constant and $f(x)$ is given in Problem 11.]

13. Suppose that $p(x)$ is a polynomial, and that $k \in \mathbb{R}$ is a constant. Suppose further that $A < B$ are consecutive roots of the equation $p(x) = 0$.
 a) Write $p(x) = (x - A)^m (x - B)^n q(x)$, where $q(A) \neq 0$ and $q(B) \neq 0$. Prove that if we write $p'(x) = (x - A)^{m-1}(x - B)^{n-1} r(x)$, then $r(A)$ and $r(B)$ have opposite signs.
 b) Hence, or otherwise, prove that there is a root of the equation $p'(x) + kp(x) = 0$ in the interval $[A, B]$.

14. Suppose that $f''(a)$ exists. Prove that $\displaystyle\lim_{h \to 0} \frac{f(a + h) - 2f(a) + f(a - h)}{h^2} = f''(a)$.

15. Suppose that a function $f(x)$ is differentiable at every $x \in [A, B]$. Prove that $f'(x)$ takes every value between $f'(A)$ and $f'(B)$.