FIRST YEAR CALCULUS

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Chapter 9

THE DEFINITE INTEGRAL

9.1. Finite Sums

We begin by considering a simple example.

EXAMPLE 9.1.1. Consider the expression

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{57}.$$

It is convenient to have a good notation. We may perhaps write

$$\sum_{i=2}^{57} \frac{1}{i}$$

instead, if we recognize that all the numbers in the sum are of the form $1/i$, where $i = 2, 3, 4, \ldots, 57$. Note that

$$\sum_{i=2}^{57} \frac{1}{i} = \sum_{i=1}^{56} \frac{1}{i+1},$$

so that we can vary the range of summation if we are prepared to vary what we are summing over. On the other hand, note that

$$\sum_{i=2}^{57} \frac{1}{i} = \sum_{j=2}^{57} \frac{1}{j},$$

so that i and j are "dummy" variables only.

DEFINITION. Suppose that $m, n \in \mathbb{Z}$ and $m < n$. Then we write

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \ldots + a_{n-1} + a_n.$$

This is called a finite sum or a finite series.

EXAMPLE 9.1.2. We have

$$\sum_{i=1}^{5} \frac{1}{1+i^2} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26}.$$

EXAMPLE 9.1.3. We have

$$\sum_{j=3}^{7} \frac{1}{j(j+1)} = \sum_{i=2}^{6} \frac{1}{(i+1)(i+2)} = \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \frac{1}{56}.$$

The following result is a simple consequence of the usual rules of addition and multiplication of real numbers.

PROPOSITION 9A. *Suppose that $m, n \in \mathbb{Z}$ and $m < n$. Suppose further that $c \in \mathbb{R}$. Then*

$$\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i \qquad \text{and} \qquad \sum_{i=m}^{n} ca_i = c \sum_{i=m}^{n} a_i.$$

EXAMPLE 9.1.4. Suppose that $n \in \mathbb{N}$. Consider the sum

$$S_n = \sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n.$$

Note that

$$\begin{aligned} 2S_n &= (1 + 2 + 3 + \ldots + n) + (n + (n-1) + (n-2) + \ldots + 1) \\ &= (1 + n) + (2 + (n-1)) + (3 + (n-2)) + \ldots + (n+1) = n(n+1), \end{aligned}$$

so that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

EXAMPLE 9.1.5. Suppose that $n \in \mathbb{N}$. Consider the sum

$$T_n = \sum_{i=1}^{n} i^2 = 1 + 4 + 9 + \ldots + n^2.$$

For every $i = 1, 2, 3, \ldots, n$, we have

$$(i+1)^3 - i^3 = 3i^2 + 3i + 1,$$

so that

$$\sum_{i=1}^{n}((i+1)^3 - i^3) = 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1.$$

In other words, we have

$$(n+1)^3 - 1 = 3T_n + 3S_n + n.$$

It follows that

$$\begin{aligned} T_n &= \frac{(n+1)^3-1}{3} - S_n - \frac{n}{3} = \frac{(n+1)^3-(n+1)}{3} - S_n = \frac{(n+1)^3-(n+1)}{3} - \frac{n(n+1)}{2} \\ &= \frac{(n+1)(2(n+1)^2-2-3n)}{6} = \frac{(n+1)(2n^2+n)}{6} = \frac{n(n+1)(2n+1)}{6}, \end{aligned}$$

so that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}. \tag{1}$$

9.2. An Example

Consider the function $f(x) = x^2$ in the interval $[-1,2]$. Suppose that we wish to find the area bounded by the curve $y = f(x)$ and the lines $y = 0$, $x = -1$ and $x = 2$ (the reader should start drawing a diagram). Unfortunately, our knowledge on areas is restricted to simple geometric shapes, and the area in question cannot be calculated by a simple area formula. So let us try some approximations.

Let us first break the interval $[-1,2]$ into shorter intervals in some arbitrary fashion, say

$$[-1,-\tfrac{1}{2}],[-\tfrac{1}{2},\tfrac{1}{4}],[\tfrac{1}{4},\tfrac{5}{4}],[\tfrac{5}{4},2]$$

(the reader should draw all the rectangles discussed below).

Consider first the interval $[-1,-\frac{1}{2}]$. We approximate the area bounded by the curve $y = f(x)$ and the lines $y = 0$, $x = -1$ and $x = -\frac{1}{2}$ by rectangles with base $[-1,-\frac{1}{2}]$ on the line $y = 0$. Note that

$$\min_{x\in[-1,-1/2]} f(x) = f\left(-\frac{1}{2}\right) = \frac{1}{4} \qquad \text{and} \qquad \max_{x\in[-1,-1/2]} f(x) = f(-1) = 1.$$

If we draw a rectangle with height $1/4$, then this rectangle has area $1/8$, clearly an under-estimate. If we draw a rectangle with height 1, then this rectangle has area $1/2$, clearly an over-estimate.

Consider next the interval $[-\frac{1}{2},\frac{1}{4}]$. We approximate the area bounded by the curve $y = f(x)$ and the lines $y = 0$, $x = -\frac{1}{2}$ and $x = \frac{1}{4}$ by rectangles with base $[-\frac{1}{2},\frac{1}{4}]$ on the line $y = 0$. Note that

$$\min_{x\in[-1/2,1/4]} f(x) = f(0) = 0 \qquad \text{and} \qquad \max_{x\in[-1/2,1/4]} f(x) = f\left(-\frac{1}{2}\right) = \frac{1}{4}.$$

If we draw a rectangle with height 0, then this rectangle has area 0, clearly an under-estimate. If we draw a rectangle with height $1/4$, then this rectangle has area $3/16$, clearly an over-estimate.

Consider next the interval $[\frac{1}{4}, \frac{5}{4}]$. We approximate the area bounded by the curve $y = f(x)$ and the lines $y = 0$, $x = \frac{1}{4}$ and $x = \frac{5}{4}$ by rectangles with base $[\frac{1}{4}, \frac{5}{4}]$ on the line $y = 0$. Note that

$$\min_{x\in[1/4,5/4]} f(x) = f\left(\frac{1}{4}\right) = \frac{1}{16} \qquad \text{and} \qquad \max_{x\in[1/4,5/4]} f(x) = f\left(\frac{5}{4}\right) = \frac{25}{16}.$$

If we draw a rectangle with height 1/16, then this rectangle has area 1/16, clearly an under-estimate. If we draw a rectangle with height 25/16, then this rectangle has area 25/16, clearly an over-estimate.

Consider finally the interval $[\frac{5}{4}, 2]$. We approximate the area bounded by the curve $y = f(x)$ and the lines $y = 0$, $x = \frac{5}{4}$ and $x = 2$ by rectangles with base $[\frac{5}{4}, 2]$ on the line $y = 0$. Note that

$$\min_{x\in[5/4,2]} f(x) = f\left(\frac{5}{4}\right) = \frac{25}{16} \qquad \text{and} \qquad \max_{x\in[5/4,2]} f(x) = f(2) = 4.$$

If we draw a rectangle with height 25/16, then this rectangle has area 75/64, clearly an under-estimate. If we draw a rectangle with height 4, then this rectangle has area 3, clearly an over-estimate.

Now let us return to the area in question, namely the area bounded by the curve $y = f(x)$ and the lines $y = 0$, $x = -1$ and $x = 2$. If we use the smaller of the two rectangles in each instance, then we get the under-estimate

$$\frac{1}{8} + 0 + \frac{1}{16} + \frac{75}{64} = \frac{87}{64}.$$

If we use the larger of the two rectangles in each instance, then we get the over-estimate

$$\frac{1}{2} + \frac{3}{16} + \frac{25}{16} + 3 = \frac{21}{4}.$$

Clearly these are very far from the truth. This is hardly surprising, as the approximations we have made are very crude indeed.

9.3. The Riemann Integral

To get further, we need to be more systematic in our treatment. The following example illustrates the key points of our technique.

EXAMPLE 9.3.1. Consider the function $f(x) = x^2$ in the interval $[0, 1]$. Suppose that we wish to find the area, A say, bounded by the curve $y = f(x)$ and the lines $y = 0$, $x = 0$ and $x = 1$ (the reader should again start drawing a diagram). Let us consider a dissection

$$\Delta_n : \ 0 = x_0 < x_1 < x_2 < \ldots < x_n = 1$$

of the interval $[0, 1]$, where $x_i = i/n$ for every $i = 0, 1, 2, \ldots, n$. For every subinterval $[x_{i-1}, x_i]$, where $i = 1, 2, \ldots, n$, we have

$$\min_{x\in[x_{i-1},x_i]} f(x) = \min_{\frac{i-1}{n}\le x\le\frac{i}{n}} x^2 = f\left(\frac{i-1}{n}\right) = \frac{(i-1)^2}{n^2}$$

and

$$\max_{x\in[x_{i-1},x_i]} f(x) = \max_{\frac{i-1}{n}\le x\le\frac{i}{n}} x^2 = f\left(\frac{i}{n}\right) = \frac{i^2}{n^2}.$$

It follows that the area, A_i say, bounded by the curve $y = f(x)$ and the lines $y = 0$, $x = x_{i-1} = (i-1)/n$ and $x = x_i = i/n$ can be approximated below by the area of a rectangle of height $(i-1)^2/n^2$ and approximated above by the area of a rectangle of height i^2/n^2; in each case, the base of the rectangle has length $x_i - x_{i-1} = 1/n$. Hence

$$(x_i - x_{i-1}) \min_{x\in[x_{i-1},x_i]} f(x) \le A_i \le (x_i - x_{i-1}) \max_{x\in[x_{i-1},x_i]} f(x);$$

more precisely,

$$\frac{(i-1)^2}{n^3} \le A_i \le \frac{i^2}{n^3}.$$

Clearly

$$A = \sum_{i=1}^n A_i.$$

Now write

$$s(f,\Delta_n) = \sum_{i=1}^n (x_i - x_{i-1}) \min_{x\in[x_{i-1},x_i]} f(x) = \sum_{i=1}^n \frac{(i-1)^2}{n^3}$$

and

$$S(f,\Delta_n) = \sum_{i=1}^n (x_i - x_{i-1}) \max_{x\in[x_{i-1},x_i]} f(x) = \sum_{i=1}^n \frac{i^2}{n^3}.$$

Then it clearly follows that

$$s(f,\Delta_n) \le A \le S(f,\Delta_n).$$

By (1), we have

$$\sum_{i=1}^n \frac{(i-1)^2}{n^3} = \frac{1}{n^3}\sum_{i=1}^n (i-1)^2 = \frac{1}{n^3}\sum_{i=0}^{n-1} i^2 = \frac{1}{n^3}\sum_{i=1}^{n-1} i^2 = \frac{(n-1)n(2n-1)}{6n^3}$$

and

$$\sum_{i=1}^n \frac{i^2}{n^3} = \frac{1}{n^3}\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6n^3}.$$

Hence

$$\frac{(n-1)n(2n-1)}{6n^3} \le A \le \frac{n(n+1)(2n+1)}{6n^3}.$$

Suppose now that n is very large. In other words, suppose that $n \to \infty$. Then

$$\frac{(n-1)n(2n-1)}{6n^3} \to \frac{1}{3} \qquad \text{and} \qquad \frac{n(n+1)(2n+1)}{6n^3} \to \frac{1}{3}.$$

It follows that we must have $A = 1/3$. Of course, we know that

$$\int_0^1 x^2 \,\mathrm{d}x = \frac{1}{3}.$$

Suppose next that instead of approximating each A_i by the two rectangles with heights

$$\min_{x\in[x_{i-1},x_i]} f(x) = \frac{(i-1)^2}{n^2} \qquad \text{and} \qquad \max_{x\in[x_{i-1},x_i]} f(x) = \frac{i^2}{n^2},$$

we simply choose some $\xi_i \in [x_{i-1}, x_i]$ and approximate A_i by a rectangle of height $f(\xi_i) = \xi_i^2$. Then we have the approximation

$$(x_i - x_{i-1})f(\xi_i) = \frac{\xi_i^2}{n}$$

for A_i and the approximation

$$\sum_{i=1}^{n}(x_i - x_{i-1})f(\xi_i) = \sum_{i=1}^{n}\frac{\xi_i^2}{n}$$

for A. Clearly

$$\frac{(i-1)^2}{n^2} \le \xi_i^2 \le \frac{i^2}{n^2},$$

so that

$$s(f,\Delta_n) \le \sum_{i=1}^{n}\frac{\xi_i^2}{n} \le S(f,\Delta_n).$$

It follows that

$$\frac{(n-1)n(2n-1)}{6n^3} \le \sum_{i=1}^{n}(x_i - x_{i-1})f(\xi_i) \le \frac{n(n+1)(2n+1)}{6n^3}.$$

Hence, for very large n,

$$\sum_{i=1}^{n}(x_i - x_{i-1})f(\xi_i)$$

is a good approximation for A.

DEFINITION. Suppose that $f(x)$ is a continuous function in the closed interval $[A, B]$, where $A, B \in \mathbb{R}$ and $A < B$. Suppose further that

$$\Delta: \ A = x_0 < x_1 < x_2 < \ldots < x_n = B$$

is a dissection of the interval $[A, B]$. Then the sum

$$s(f,\Delta) = \sum_{i=1}^{n}(x_i - x_{i-1}) \min_{x\in[x_{i-1},x_i]} f(x)$$

is called the lower Riemann sum of $f(x)$ corresponding to the dissection Δ, and the sum

$$S(f,\Delta) = \sum_{i=1}^{n}(x_i - x_{i-1}) \max_{x\in[x_{i-1},x_i]} f(x)$$

is called the upper Riemann sum of $f(x)$ corresponding to the dissection Δ. Suppose further that for every $i = 1, \ldots, n$, we have $\xi_i \in [x_{i-1}, x_i]$. Then the sum

$$\sum_{i=1}^{n}(x_i - x_{i-1})f(\xi_i)$$

is called a Riemann sum of $f(x)$ corresponding to the dissection Δ.

REMARKS. (1) It is clear that

$$\min_{x\in[x_{i-1},x_i]} f(x) \le f(\xi_i) \le \max_{x\in[x_{i-1},x_i]} f(x).$$

It follows that every Riemann sum is bounded below by the corresponding lower Riemann sum and bounded above by the corresponding upper Riemann sum; in other words,

$$s(f,\Delta) \le \sum_{i=1}^{n}(x_i - x_{i-1})f(\xi_i) \le S(f,\Delta).$$

(2) It can be shown that for any two dissections Δ' and Δ'' of the closed interval $[A,B]$, we have

$$s(f,\Delta') \le S(f,\Delta'');$$

in other words, a lower Riemann sum can never exceed an upper Riemann sum.

(3) Note that we have restricted our attention to continuous functions in the closed interval $[A,B]$. This is in fact unnecessary. It is enough to assume that the function $f(x)$ is bounded in the closed interval $[A,B]$. However, the definition of the lower and upper Riemann sums need to be slightly modified. We shall discuss this more general setting in Section 9.7.

DEFINITION. We say that

$$\int_A^B f(x)\,\mathrm{d}x = L$$

if, given any $\epsilon > 0$, there exists a dissection Δ of $[A,B]$ such that

$$L - \epsilon < s(f,\Delta) \le S(f,\Delta) < L + \epsilon.$$

In this case, we say that the continuous function $f(x)$ is Riemann integrable over the closed interval $[A,B]$ with integral L.

REMARK. In other words, if the lower Riemann sums and upper Riemann sums can get arbitrarily close, then their common value is the integral of the function.

We state here the following important result. For a formal proof, see Section 9.7.

PROPOSITION 9B. *Suppose that $f(x)$ is a continuous function in the closed interval $[A,B]$, where $A,B\in\mathbb{R}$ and $A<B$. Then $f(x)$ is Riemann integrable over $[A,B]$.*

EXAMPLE 9.3.2. Consider the function $f(x) = \sin x$ in the closed interval $[0,\pi/2]$. Suppose that

$$\Delta: \ 0 = x_0 < x_1 < x_2 < \ldots < x_n = \frac{\pi}{2}$$

is a dissection of the interval $[0,\pi/2]$, where

$$x_i = \frac{i\pi}{2n}, \quad i = 0,1,2,\ldots,n.$$

Since $f(x) = \sin x$ is increasing in $[0,\pi/2]$, it follows that

$$\min_{x\in[x_{i-1},x_i]} f(x) = f(x_{i-1}) = \sin\frac{(i-1)\pi}{2n} \qquad \text{and} \qquad \max_{x\in[x_{i-1},x_i]} f(x) = f(x_i) = \sin\frac{i\pi}{2n}.$$

Hence

$$s(f,\Delta)=\sum_{i=1}^n (x_i-x_{i-1})\min_{x\in[x_{i-1},x_i]} f(x)=\frac{\pi}{2n}\sum_{i=1}^n \sin\frac{(i-1)\pi}{2n}=\frac{\pi}{2n}\sum_{i=0}^{n-1}\sin\frac{i\pi}{2n}=\frac{\pi}{2n}\sum_{i=1}^{n-1}\sin\frac{i\pi}{2n}$$

and

$$S(f,\Delta)=\sum_{i=1}^n (x_i-x_{i-1})\max_{x\in[x_{i-1},x_i]} f(x)=\frac{\pi}{2n}\sum_{i=1}^n \sin\frac{i\pi}{2n}.$$

Next, note that

$$\sum_{i=1}^n \sin\frac{i\pi}{2n}=\sum_{i=1}^n \frac{\cos((i-\frac{1}{2})\frac{\pi}{2n})-\cos((i+\frac{1}{2})\frac{\pi}{2n})}{2\sin\frac{\pi}{4n}}=\frac{\cos\frac{\pi}{4n}-\cos(\frac{\pi}{2}+\frac{\pi}{4n})}{2\sin\frac{\pi}{4n}}.$$

Similarly,

$$\sum_{i=1}^{n-1} \sin\frac{i\pi}{2n}=\frac{\cos\frac{\pi}{4n}-\cos(\frac{\pi}{2}-\frac{\pi}{4n})}{2\sin\frac{\pi}{4n}}.$$

It follows that as $n\to\infty$, we have

$$s(f,\Delta)=\frac{\pi}{2n}\left(\frac{\cos\frac{\pi}{4n}-\cos(\frac{\pi}{2}-\frac{\pi}{4n})}{2\sin\frac{\pi}{4n}}\right)\to\cos 0-\cos\frac{\pi}{2}=1,$$

and

$$S(f,\Delta)=\frac{\pi}{2n}\left(\frac{\cos\frac{\pi}{4n}-\cos(\frac{\pi}{2}+\frac{\pi}{4n})}{2\sin\frac{\pi}{4n}}\right)\to\cos 0-\cos\frac{\pi}{2}=1.$$

Hence

$$\int_0^{\pi/2}\sin x\,\mathrm{d}x=1.$$

9.4. Antiderivatives

Our aim is to relate our definition of the Riemann integral to something more familiar. The first step in this direction involves the study of antiderivatives or indefinite integrals.

Definition. A function $F(x)$ is called an antiderivative or indefinite integral of a function $f(x)$ in an interval I if $F'(x)=f(x)$ for every $x\in I$.

Example 9.4.1. Suppose that $f(x)=3x^2$. Then for any $C\in\mathbb{R}$, the function $F(x)=x^3+C$ is an antiderivative of $f(x)$ on any interval. It follows that there are infinitely many antiderivatives that differ by constants.

The next result shows that there are no more. The proof, which depends on the Mean value theorem, is given in Section 9.7.

PROPOSITION 9C. *Suppose that the function $F(x)$ is an antiderivative of a function $f(x)$ in an interval I. Then every antiderivative of $f(x)$ is of the form $F(x)+C$, where $C\in\mathbb{R}$ is a constant.*

EXAMPLE 9.4.2. The following table of antiderivatives can be checked for appropriate intervals I:

$f(x)$	$F(x)$	$f(x)$	$F(x)$
0	C	e^x	$e^x + C$
$\cos x$	$\sin x + C$	$\sin x$	$-\cos x + C$
$\sec^2 x$	$\tan x + C$	$\csc^2 x$	$-\cot x + C$
$\sec x \tan x$	$\sec x + C$	$\csc x \cot x$	$-\csc x + C$
$\tan x$	$\log \lvert \sec x \rvert + C$	$\cot x$	$-\log \lvert \csc x \rvert + C$
$\sec x$	$\log \lvert \sec x + \tan x \rvert + C$	$\csc x$	$\log \lvert \csc x - \cot x \rvert + C$
$(n+1)x^n \ (n \neq -1)$	$x^{n+1} + C$	x^{-1}	$\log \lvert x \rvert + C$

The next result is crucial in the calculation of antiderivatives. The proof is straightforward, in view of Proposition 3B.

PROPOSITION 9D. *Suppose that the functions $F(x)$ and $G(x)$ are antiderivatives of functions $f(x)$ and $g(x)$ respectively in an interval I. Suppose further that $c \in \mathbb{R}$. Then*
(a) $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$ in I; and
(b) $cF(x)$ is an antiderivative of $cf(x)$ in I.

EXAMPLE 9.4.3. Suppose that $f(x) = x^2 + 2\sin x$. We can write

$$f(x) = \frac{1}{3}g(x) + 2h(x),$$

where

$$g(x) = 3x^2 \qquad \text{and} \qquad h(x) = \sin x.$$

From the table in Example 9.4.2 and with $C = 0$, the functions

$$G(x) = x^3 \qquad \text{and} \qquad H(x) = -\cos x$$

are antiderivatives of $g(x)$ and $h(x)$ respectively in any interval. It follows from Proposition 9D that the function

$$F(x) = \frac{1}{3}G(x) + 2H(x) = \frac{x^3}{3} - 2\cos x$$

is an antiderivative of $f(x)$ in any interval, so that it follows from Proposition 9C that every antiderivative of $f(x)$ in any interval is of the form

$$\frac{x^3}{3} - 2\cos x + C,$$

where $C \in \mathbb{R}$.

For the sake of convenience, we shall denote any antiderivative of a function $f(x)$ by

$$\int f(x)\,dx.$$

Also, we may choose to omit reference to the interval I in question, with the understanding that an appropriate interval I has been chosen.

9.5. Fundamental Theorems of the Integral Calculus

In this section, we shall first discuss a relationship between the Riemann integral and antiderivatives. This relationship enables us to calculate the Riemann integral by simply finding an antiderivative of the given function. In Section 9.7, we shall establish the following important result.

PROPOSITION 9E. (FUNDAMENTAL THEOREM OF THE INTEGRAL CALCULUS) *Suppose that $f(x)$ is a continuous function in the closed interval $[A,B]$, where $A,B \in \mathbb{R}$ and $A < B$. Suppose further that the function $F(x)$ is an antiderivative of $f(x)$ in $[A,B]$. Then*

$$\int_A^B f(x)\,\mathrm{d}x = F(B) - F(A).$$

EXAMPLE 9.5.1. We have

$$\int_\pi^{2\pi} (x^2 + 2\sin x)\,\mathrm{d}x = \left[\frac{x^3}{3} - 2\cos x\right]_\pi^{2\pi} = \left(\frac{8\pi^3}{3} - 2\cos 2\pi\right) - \left(\frac{\pi^3}{3} - 2\cos\pi\right) = \frac{7\pi^3}{3} - 4.$$

EXAMPLE 9.5.2. We have

$$\int_0^\pi \sqrt{1-\cos x}\sin x\,\mathrm{d}x = \left[\frac{2}{3}(1-\cos x)^{3/2}\right]_0^\pi = \frac{2}{3}(1-\cos\pi)^{3/2} - \frac{2}{3}(1-\cos 0)^{3/2} = \frac{4\sqrt{2}}{3}.$$

EXAMPLE 9.5.3. We have

$$\int_3^4 \frac{x}{\sqrt{x-2}}\,\mathrm{d}x = \left[\frac{2}{3}(x-2)^{3/2} + 4(x-2)^{1/2}\right]_3^4 = \frac{16\sqrt{2}-14}{3}.$$

EXAMPLE 9.5.4. We have

$$\int_0^{\pi/2} \sin^3 x\cos^3 x\,\mathrm{d}x = \left[\frac{\sin^4 x}{4} - \frac{\sin^6 x}{6}\right]_0^{\pi/2} = \frac{1}{12}.$$

EXAMPLE 9.5.5. The argument

$$\int_{-1}^1 \frac{1}{x^2}\,\mathrm{d}x = \left[-\frac{1}{x}\right]_{-1}^1 = -1 - 1 = -2$$

is clearly wrong, since the curve is never below the line $y = 0$ between $x = -1$ and $x = 1$. Note that the function $1/x^2$ is not continuous in the interval $[-1,1]$, so that the Fundamental theorem of the integral calculus does not apply.

Riemann integrals can, in a certain sense, be regarded as antiderivatives. The following result is sometimes known as the second Fundamental theorem of the integral calculus.

PROPOSITION 9F. *Suppose that $f(x)$ is a continuous function in the closed interval $[A,B]$, where $A,B \in \mathbb{R}$ and $A < B$. Then the function*

$$F(x) = \int_A^x f(t)\,\mathrm{d}t$$

is continuous in the closed interval $[A,B]$ and differentiable for every $x \in (A,B)$, with $F'(x) = f(x)$.

REMARK. In some treatments, Propositions 9E and 9F are given in reverse order.

EXAMPLE 9.5.6. We have

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \sqrt{1+t^2}\,\mathrm{d}t = \sqrt{1+x^2}.$$

EXAMPLE 9.5.7. We have

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_x^1 \sqrt{1+t^2}\,\mathrm{d}t = -\frac{\mathrm{d}}{\mathrm{d}x}\int_1^x \sqrt{1+t^2}\,\mathrm{d}t = -\sqrt{1+x^2}.$$

EXAMPLE 9.5.8. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_{2x}^{x^2} \sin(4t^3+3)\,\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_C^{x^2} \sin(4t^3+3)\,\mathrm{d}t + \int_{2x}^{C} \sin(4t^3+3)\,\mathrm{d}t\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\int_C^{x^2} \sin(4t^3+3)\,\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}x}\int_{2x}^{C} \sin(4t^3+3)\,\mathrm{d}t\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\int_C^{x^2} \sin(4t^3+3)\,\mathrm{d}t - \frac{\mathrm{d}}{\mathrm{d}x}\int_C^{2x} \sin(4t^3+3)\,\mathrm{d}t.\end{aligned}$$

Writing $y=x^2$ and using the Chain rule, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_C^{x^2} \sin(4t^3+3)\,\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x}\int_C^{y} \sin(4t^3+3)\,\mathrm{d}t &= \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}y}\int_C^{y} \sin(4t^3+3)\,\mathrm{d}t\\ &= 2x\sin(4y^3+3) = 2x\sin(4x^6+3).\end{aligned}$$

Writing $u=2x$ and using the Chain rule, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_C^{2x} \sin(4t^3+3)\,\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x}\int_C^{u} \sin(4t^3+3)\,\mathrm{d}t &= \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}u}\int_C^{u} \sin(4t^3+3)\,\mathrm{d}t\\ &= 2\sin(4u^3+3) = 2\sin(32x^3+3).\end{aligned}$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{2x}^{x^2} \sin(4t^3+3)\,\mathrm{d}t = 2x\sin(4x^6+3) - 2\sin(32x^3+3).$$

The next two results can be considered to be simple consequences of the Fundamental theorems of the integral calculus. In Section 9.7, we shall discuss how we can establish more general versions of these two results.

PROPOSITION 9G. *Suppose that $f(x)$ is a continuous function in the closed interval $[A,B]$, where $A,B\in\mathbb{R}$ and $A<B$. Suppose further that $C\in[A,B]$. Then*

$$\int_A^B f(x)\,\mathrm{d}x = \int_A^C f(x)\,\mathrm{d}x + \int_C^B f(x)\,\mathrm{d}x.$$

PROPOSITION 9H. *Suppose that $f(x)$ and $g(x)$ are continuous function in the closed interval $[A,B]$, where $A,B\in\mathbb{R}$ and $A<B$. Then*

$$\int_A^B (f(x)+g(x))\,\mathrm{d}x = \int_A^B f(x)\,\mathrm{d}x + \int_A^B g(x)\,\mathrm{d}x.$$

Furthermore, for every real number $c\in\mathbb{R}$, we have

$$\int_A^B cf(x)\,\mathrm{d}x = c\int_A^B f(x)\,\mathrm{d}x.$$

The following result gives some very crude bound for the Riemann integral.

PROPOSITION 9J. *Suppose that $f(x)$ is a continuous function in the closed interval $[A,B]$, where $A,B\in\mathbb{R}$ and $A<B$. Suppose further that $m\le f(x)\le M$ for every $x\in[A,B]$. Then*

$$m(B-A)\le\int_A^B f(x)\,\mathrm{d}x\le M(B-A).$$

9.6. Average Values of Functions

Suppose that the function $f(x)$ is non-negative and continuous in the closed interval $[A,B]$. Then the Riemann integral

$$\int_A^B f(x)\,\mathrm{d}x$$

exists and represents the area bounded by the curve $y=f(x)$ and the lines $y=0$, $x=A$ and $x=B$. Consider a rectangle with its base on the x-axis between $x=A$ and $x=B$ and with the same area as the integral. Then its height

$$\frac{1}{B-A}\int_A^B f(x)\,\mathrm{d}x$$

must represent the average value of the function $f(x)$ in the interval $[A,B]$.

Of course, the restriction that $f(x)$ is non-negative is not necessary and can be removed.

EXAMPLE 9.6.1. The average value of the function $\sin x$ in the interval $[0,2\pi]$ is

$$\frac{1}{2\pi}\int_0^{2\pi}\sin x\,\mathrm{d}x=0.$$

EXAMPLE 9.6.2. The average value of the function $\sin x$ in the interval $[0,\pi]$ is

$$\frac{1}{\pi}\int_0^{\pi}\sin x\,\mathrm{d}x=\frac{2}{\pi}.$$

EXAMPLE 9.6.3. The average value of the function x^2 in the interval $[0,2]$ is

$$\frac{1}{2}\int_0^{2}x^2\,\mathrm{d}x=\frac{4}{3}.$$

EXAMPLE 9.6.4. The average value of the function $\sin^2 x$ in the interval $[0, 2\pi]$ is

$$\frac{1}{2\pi} \int_0^{2\pi} \sin^2 x \, \mathrm{d}x = \frac{1}{2}.$$

In Section 9.7, we shall establish the following result which shows that the mean value is attained by the function in the interval.

PROPOSITION 9K. (MEAN VALUE THEOREM FOR RIEMANN INTEGRALS) *Suppose that $f(x)$ is a continuous function in the closed interval $[A, B]$, where $A, B \in \mathbb{R}$ and $A < B$. Then there exists $c \in (A, B)$ such that*

$$\frac{1}{B - A} \int_A^B f(x) \, \mathrm{d}x = f(c).$$

9.7. Further Discussion

Here we shall study the Riemann integral in a more general setting. Suppose that $f(x)$ is a real valued function bounded in the closed interval $[A, B]$, where $A, B \in \mathbb{R}$ and $A < B$. Suppose further that

$$\Delta : \; A = x_0 < x_1 < x_2 < \ldots < x_n = B$$

is a dissection of the interval $[A, B]$. Then the sum

$$s(f, \Delta) = \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{x \in [x_{i-1}, x_i]} f(x)$$

is called the lower Riemann sum of $f(x)$ corresponding to the dissection Δ, and the sum

$$S(f, \Delta) = \sum_{i=1}^{n} (x_i - x_{i-1}) \sup_{x \in [x_{i-1}, x_i]} f(x)$$

is called the upper Riemann sum of $f(x)$ corresponding to the dissection Δ. Suppose further that for every $i = 1, \ldots, n$, we have $\xi_i \in [x_{i-1}, x_i]$. Then the sum

$$\sum_{i=1}^{n} (x_i - x_{i-1}) f(\xi_i)$$

is called a Riemann sum of $f(x)$ corresponding to the dissection Δ.

Note that it is important to use the infimum and supremum instead of minimum and maximum, as the latter may not exist, while the former always exist, since the function $f(x)$ is bounded in $[A, B]$. In fact, if the function $f(x)$ is continuous in the interval $[A, B]$, then $f(x)$ is bounded in $[A, B]$ in view of Proposition 4E, and attains a minimum and maximum in any closed subinterval of $[A, B]$ in view of Proposition 4C. This means that the infimum is a minimum, while the supremum is a maximum. It follows that the above definitions represent a generalization of our earlier definitions.

The following three results are easy to establish, and are left as exercises for the interested reader.

PROPOSITION 9L. *Suppose that $f(x)$ is a real valued function bounded in the closed interval $[A, B]$, where $A, B \in \mathbb{R}$ and $A < B$. Then for any dissection Δ of the interval $[A, B]$, we have*

$$s(f, \Delta) \le S(f, \Delta).$$

PROPOSITION 9M. *Suppose that $f(x)$ is a real valued function bounded in the closed interval $[A,B]$, where $A,B \in \mathbb{R}$ and $A < B$. Suppose further that Δ and Δ' are two dissections of the interval $[A,B]$ satisfying $\Delta \subseteq \Delta'$; in other words, every dissection point of Δ is also a dissection point of Δ'. Then*

$$s(f,\Delta) \le s(f,\Delta') \qquad \text{and} \qquad S(f,\Delta') \le S(f,\Delta).$$

Combining the two results above, we have the following result alluded to in our earlier discussion.

PROPOSITION 9N. *Suppose that $f(x)$ is a real valued function bounded in the closed interval $[A,B]$, where $A,B \in \mathbb{R}$ and $A < B$. Then for any dissections Δ' and Δ'' of the interval $[A,B]$, we have*

$$s(f,\Delta') \le S(f,\Delta'').$$

In other words, a lower Riemann sum can never exceed an upper Riemann sum.

Definition. Suppose that $f(x)$ is a real valued function bounded in the closed interval $[A,B]$, where $A,B \in \mathbb{R}$ and $A < B$. The quantity

$$L^+(f) = \inf_{\Delta} S(f,\Delta) \tag{2}$$

is called the upper integral of $f(x)$ in $[A,B]$, and the quantity

$$L^-(f) = \sup_{\Delta} s(f,\Delta) \tag{3}$$

is called the lower integral of $f(x)$ in $[A,B]$. Here the infimum and supremum are taken over all dissections Δ of the interval $[A,B]$. Furthermore, if $L^+(f) = L^-(f)$, then we say that the function $f(x)$ is Riemann integrable over the interval $[A,B]$, and denote by

$$L = \int_A^B f(x)\,\mathrm{d}x$$

the common value of the upper and lower integrals of $f(x)$ in $[A,B]$.

Note that the existence of the upper and lower integrals are guaranteed by the boundedness of the function $f(x)$ in the interval $[A,B]$.

Example 9.7.1. It is not easy to find a function that is not Riemann integrable. Here, we shall give one, but the proof depends on some rather deep result on rational and irrational numbers. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

It is well known that in any open interval, there are rational numbers and irrational numbers. It follows that in any interval $[\alpha,\beta]$, where $\alpha < \beta$, we have

$$\inf_{x\in[\alpha,\beta]} f(x) = 0 \qquad \text{and} \qquad \sup_{x\in[\alpha,\beta]} f(x) = 1.$$

It follows that for every dissection Δ of $[0,1]$, we have

$$s(f,\Delta) = 0 \qquad \text{and} \qquad S(f,\Delta) = 1.$$

Hence $f(x)$ is not Riemann integrable over the closed interval $[0,1]$.

Remark. Note that the Riemann integral never exceeds any upper Riemann sum and is never less than any lower Riemann sum. A consequence of this simple observation is Proposition 9J.

We now need to show that our definition of Riemann integrability here agrees with our earlier definition in the case of continuous functions. We first establish the following result.

PROPOSITION 9P. *Suppose that $f(x)$ is a real valued function bounded in the closed interval $[A,B]$, where $A,B \in \mathbb{R}$ and $A<B$. Then $f(x)$ is Riemann integrable over the interval $[A,B]$ if and only if, given any $\epsilon>0$, there exists a dissection Δ of $[A,B]$ such that*

$$S(f,\Delta)-s(f,\Delta)<\epsilon.$$

Proof. Suppose first of all that $f(x)$ is Riemann integrable over the interval $[A,B]$. Then

$$L^+(f)=L^-(f)=L.$$

Let $\epsilon>0$ be given. In view of (2) and (3), there exist dissections Δ' and Δ'' of $[A,B]$ such that

$$S(f,\Delta')<L^+(f)+\frac{\epsilon}{2} \qquad \text{and} \qquad s(f,\Delta')>L^-(f)-\frac{\epsilon}{2}.$$

Let $\Delta=\Delta'\cup\Delta''$; in other words, Δ contains precisely all the dissection points of both Δ' and Δ''. Then it follows from Proposition 9M that

$$S(f,\Delta)\le S(f,\Delta') \qquad \text{and} \qquad s(f,\Delta)\ge s(f,\Delta'').$$

Combining the above and noting Proposition 9L, we have

$$L-\frac{\epsilon}{2}<s(f,\Delta)\le S(f,\Delta)<L+\frac{\epsilon}{2},$$

so that $S(f,\Delta)-s(f,\Delta)<\epsilon$.

On the other hand, it is clear from Proposition 9N that $L^-(f)\le L^+(f)$. Suppose on the contrary that $f(x)$ is not Riemann integrable over the interval $[A,B]$. Then $L^-(f)\ne L^+(f)$. Let

$$\epsilon=L^+(f)-L^-(f)>0.$$

For every dissection Δ of $[A,B]$, we have

$$s(f,\Delta)\le L^-(f) \qquad \text{and} \qquad S(f,\Delta)\ge L^+(f),$$

so that $S(f,\Delta)-s(f,\Delta)\ge\epsilon$. ○

We can also establish the following stronger versions of Propositions 9G and 9H.

PROPOSITION 9G'. *Suppose that a function $f(x)$ is Riemann integrable over the closed interval $[A,B]$, where $A,B\in\mathbb{R}$ and $A<B$. Suppose further that $C\in[A,B]$. Then $f(x)$ is Riemann integrable over the closed intervals $[A,C]$ and $[C,B]$, and*

$$\int_A^B f(x)\,\mathrm{d}x=\int_A^C f(x)\,\mathrm{d}x+\int_C^B f(x)\,\mathrm{d}x.$$

PROPOSITION 9H'. *Suppose that functions $f(x)$ and $g(x)$ are Riemann integrable over the closed interval $[A,B]$, where $A,B \in \mathbb{R}$ and $A < B$. Then the function $f(x)+g(x)$ is also Riemann integrable over $[A,B]$, and*

$$\int_A^B (f(x)+g(x))\,\mathrm{d}x = \int_A^B f(x)\,\mathrm{d}x + \int_A^B g(x)\,\mathrm{d}x.$$

Furthermore, for every real number $c \in \mathbb{R}$, the function $cf(x)$ is Riemann integrable over $[A,B]$, and

$$\int_A^B cf(x)\,\mathrm{d}x = c\int_A^B f(x)\,\mathrm{d}x.$$

The proofs of these two results are left as exercises for the interested reader. An important idea is to use Proposition 9P to establish Riemann integrability first and then the definition of the Riemann integral to establish the various identities.

Our next task is to establish the important result that continuity in the closed interval $[A,B]$ implies Riemann integrability. To do this, we need to introduce the idea of uniformity. We first establish the following intermediate result.

PROPOSITION 9Q. *Suppose that $f(x)$ is a continuous function in the closed interval $[A,B]$, where $A,B \in \mathbb{R}$ and $A < B$. Then given any $\epsilon > 0$, there is a dissection $\Delta :\ A = x_0 < x_1 < x_2 < \ldots < x_n = B$ of the interval $[A,B]$ such that for every $i = 1,\ldots,n$, we have*

$$\sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x) < \frac{\epsilon}{B-A}.$$

Proof. Let $\epsilon > 0$ be given. We shall say that a subinterval $[\alpha,\beta]$ of the interval $[A,B]$ is "good" if there exists a dissection

$$\Delta' :\ \alpha = y_0 < y_1 < y_2 < \ldots < y_m = \beta$$

of the interval $[\alpha,\beta]$ such that for every $j = 1,\ldots,m$, we have

$$\sup_{x\in[y_{j-1},y_j]} f(x) - \inf_{x\in[y_{j-1},y_j]} f(x) < \frac{\epsilon}{B-A}.$$

Our task is therefore to show that the interval $[A,B]$ is good. Suppose that it is not. We bisect the interval $[A,B]$, and let C denote its midpoint. Then at least one of the two subintervals $[A,C]$ and $[C,B]$ is not good. Let this be denoted by $[a_1,b_1]$, choosing one subinterval if neither is good. We now bisect the interval $[a_1,b_1]$ to obtain a subinterval $[a_2,b_2]$ which is not good, and continue this process. We therefore have two sequences

$$a_1 \le a_2 \le a_3 \le \ldots \qquad \text{and} \qquad \ldots \le b_3 \le b_2 \le b_1$$

which clearly converge to a common value $\xi \in [A,B]$. Since $f(x)$ is continuous at $x = \xi$, there exists $\delta > 0$ such that

$$\sup_{x\in(\xi-\delta,\xi+\delta)} f(x) - \inf_{x\in(\xi-\delta,\xi+\delta)} f(x) < \frac{\epsilon}{B-A}$$

(here the interval $(\xi-\delta,\xi+\delta)$ has to be replaced by $[\xi,\xi+\delta)$ or $(\xi-\delta,\xi]$ if $\xi = A$ or $\xi = B$ respectively). On the other hand, if n is large enough, then the interval $[a_n,b_n]$ is contained in the interval $(\xi-\delta,\xi+\delta)$, and gives rise to a contradiction. ○

PROOF OF PROPOSITION 9B. Given any $\epsilon > 0$, it follows from Proposition 9Q that there exists a dissection

$$\Delta: \; A = x_0 < x_1 < x_2 < \ldots < x_n = B$$

of the interval $[A, B]$ such that for every $i = 1, \ldots, n$, we have

$$\sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x) < \frac{\epsilon}{B - A}.$$

It follows that $S(f, \Delta) - s(f, \Delta) < \epsilon$. The result now follows from Proposition 9P. ◯

The Mean value theorem for differentiation is the crux for the proof of our remaining assertions. We first use this to establish the essential uniqueness of antiderivatives.

PROOF OF PROPOSITION 9C. Suppose that $F(x)$ and $G(x)$ are two antiderivatives of the function $f(x)$ in an interval I. Write $D(x) = G(x) - F(x)$. Then

$$D'(x) = G'(x) - F'(x) = f(x) - f(x) = 0 \quad \text{for every } x \in I.$$

Suppose that $x_1, x_2 \in I$ and $x_1 < x_2$. Since $D(x)$ is differentiable for every $x \in [x_1, x_2]$, it follows from Proposition 8A that $D(x)$ is continuous in the closed interval $[x_1, x_2]$. By the Mean value theorem, there exists $\xi \in (x_1, x_2)$ such that

$$D'(\xi) = \frac{D(x_2) - D(x_1)}{x_2 - x_1}.$$

Clearly $D'(\xi) = 0$, so that

$$D(x_1) = D(x_2).$$

Note now that this argument is valid for any $x_1, x_2 \in I$. It follows that there is some constant $C \in \mathbb{R}$ such that $D(x) = C$ for every $x \in I$, whence $G(x) = F(x) + C$ for every $x \in I$. ◯

We next establish the Fundamental theorems of the integral calculus.

PROOF OF PROPOSITION 9E. By Proposition 9B, the Riemann integral exists. Write

$$L = \int_A^B f(x) \, \mathrm{d}x.$$

It follows that for every $\epsilon > 0$, there is a dissection

$$\Delta: \; A = x_0 < x_1 < x_2 < \ldots < x_n = B$$

such that

$$L - \epsilon < s(\Delta) \le S(\Delta) < L + \epsilon. \tag{4}$$

Next, note that

$$F(B) - F(A) = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})). \tag{5}$$

Since $F(x)$ is differentiable in the closed interval $[x_{i-1}, x_i]$, it follows from the Mean value theorem that there exists $\xi_i \in [x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = (x_i - x_{i-1}) F'(\xi_i) = (x_i - x_{i-1}) f(\xi_i). \tag{6}$$

Combining (5) and (6), we have

$$F(B) - F(A) = \sum_{i=1}^{n} (x_i - x_{i-1}) f(\xi_i),$$

a Riemann sum of $f(x)$ corresponding to the dissection Δ. Recall now that every Riemann sum is bounded below by the corresponding lower Riemann sum and bounded above by the corresponding upper Riemann sum, so that

$$s(\Delta) \le F(B) - F(A) \le S(\Delta). \tag{7}$$

Combining (4) and (7), we have

$$L - \epsilon < F(B) - F(A) < L + \epsilon,$$

so that $|L - (F(B) - F(A))| < \epsilon$. Since $L - (F(B) - F(A))$ is a constant and $\epsilon > 0$ is arbitrary, we must have $L - (F(B) - F(A)) = 0$. The result follows immediately. ○

Proof of Proposition 9F. Suppose first of all that $A < x < B$. Then

$$\frac{F(y) - F(x)}{y - x} = \frac{1}{y - x} \left(\int_A^y f(t)\,\mathrm{d}t - \int_A^x f(t)\,\mathrm{d}t \right) = \frac{1}{y - x} \int_x^y f(t)\,\mathrm{d}t,$$

with the convention that

$$\int_x^y f(t)\,\mathrm{d}t = -\int_y^x f(t)\,\mathrm{d}t$$

if $x > y$. We need to show that

$$\lim_{y \to x} \frac{F(y) - F(x)}{y - x} = f(x).$$

In other words, we need to show that

$$\lim_{y \to x} \frac{1}{y - x} \int_x^y f(t)\,\mathrm{d}t = f(x). \tag{8}$$

Note that

$$\left| \frac{1}{y - x} \int_x^y f(t)\,\mathrm{d}t - f(x) \right| = \left| \frac{1}{y - x} \int_x^y (f(t) - f(x))\,\mathrm{d}t \right| \le \frac{1}{|y - x|} \int_x^y |f(t) - f(x)|\,\mathrm{d}t$$

(here we have used the inequality

$$\left| \int_A^B g(x)\,\mathrm{d}x \right| \le \int_A^B |g(x)|\,\mathrm{d}x;$$

for a proof, see Problem 2). Continuity implies that for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(t) - f(x)| < \epsilon$ whenever $|t - x| < \delta$. It follows that if $|y - x| < \delta$, then

$$\frac{1}{|y - x|} \int_x^y |f(t) - f(x)|\,\mathrm{d}t < \frac{1}{|y - x|} |y - x| \epsilon = \epsilon.$$

This gives (8), and completes the proof when $A < x < B$. The cases $x = A$ and $x = B$ can be deduced with minor modifications. ○

We complete this chapter by establishing the Mean value theorem for Riemann integrals.

PROOF OF PROPOSITION 9K. By Proposition 9F, the function

$$F(x) = \int_A^x f(t)\,\mathrm{d}t$$

is continuous in the interval $[A, B]$ and differentiable for every $x \in (A, B)$, with $F'(x) = f(x)$. By the Mean value theorem, there exists $c \in (A, B)$ such that

$$F(B) - F(A) = (B - A)F'(c),$$

so that

$$\int_A^B f(t)\,\mathrm{d}t - \int_A^A f(t)\,\mathrm{d}t = (B - A)f(c).$$

Clearly the second integral vanishes, and the result follows. ○

Problems for Chapter 9

1. Calculate the integral $\displaystyle\int_0^1 x\,\mathrm{d}x$ by dissecting the interval $[0,1]$ into equal parts.

2. a) Suppose that the function $f(x)$ is continuous in the closed interval $[A,B]$. Suppose further that $f(x)\ge 0$ for every $x\in[A,B]$. Explain why

$$\int_A^B f(x)\,\mathrm{d}x \ge 0.$$

   b) Suppose that the functions $f_1(x)$ and $f_2(x)$ are continuous in the closed interval $[A,B]$. Suppose further that $f_1(x)\le f_2(x)$ for every $x\in[A,B]$. Use part (a) to show that

$$\int_A^B f_1(x)\,\mathrm{d}x \le \int_A^B f_2(x)\,\mathrm{d}x.$$

   c) Suppose that the function $g(x)$ is continuous in the closed interval $[A,B]$. Explain why

$$\left|\int_A^B g(x)\,\mathrm{d}x\right| \le \int_A^B |g(x)|\,\mathrm{d}x.$$

3. Differentiate each of the following integrals with respect to x:

   a) $\displaystyle\int_1^x t\sqrt{t^2+1}\,\mathrm{d}t$ b) $\displaystyle\int_x^4 t^2(t+1)^3\,\mathrm{d}t$ c) $\displaystyle\int_{2x+1}^{1-2x} \frac{1}{1+t^2}\,\mathrm{d}t$

4. Determine $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\int_1^{\sin x}\frac{1}{t+\sqrt{t}}\,\mathrm{d}t.$

5. Show that for every $x\in(0,1)$, we have $\displaystyle\int_{1-x}^{1+x}\frac{t-1}{t(2-t)}\,\mathrm{d}t = 0.$

6. a) Suppose that $f(x)=\sin^{-1}x+\cos^{-1}x$. Find the largest domain of $f(x)$ as a real valued function and show that $f(x)=\pi/2$ for all x in this domain.

   b) Differentiate the function $g(x)=x\sin^{-1}x+\sqrt{1-x^2}$. Hence find the integral $\displaystyle\int_0^{1/\sqrt{2}}\sin^{-1}x\,\mathrm{d}x.$

   c) Use parts (a) and (b), or otherwise, to find the area bounded by the curves $y=\sin^{-1}x$, $y=\cos^{-1}x$ and the y-axis.

7. Prove Proposition 9L.

8. Prove Proposition 9M.

9. Prove Proposition 9G'.

10. Prove Proposition 9H'.

11. Calculate the integral

$$\int_A^B x^k\,\mathrm{d}x,$$

where $k>0$ is fixed, by dissecting the interval $[A,B]$ into n parts in geometric progression, so that $A<Aq<Aq^2<\ldots<Aq^n=B$.

12. a) By using the method of Problem 11, prove that $\displaystyle\int_1^2 \frac{1}{x^2}\,\mathrm{d}x = \frac{1}{2}$.

    b) Deduce that $\displaystyle\lim_{n\to\infty} n\left(\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots + \frac{1}{(2n)^2}\right) = \frac{1}{2}$.

13. Calculate the integral $\displaystyle\int_0^\alpha \sin x\,\mathrm{d}x$ by dissecting the interval $[0, \alpha]$ into equal parts.