FIRST YEAR CALCULUS

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Chapter 12

APPLICATIONS OF INTEGRATION

12.1. Areas on the Plane

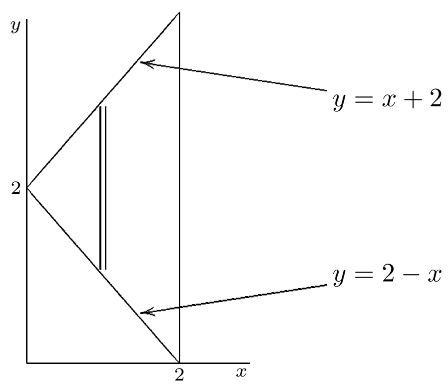
Recall that in Chapter 9, the Riemann integral

$$\int_A^B f(x)\,\mathrm{d}x$$

is formulated in terms of the area bounded by a curve $y = f(x)$ and the lines $y = 0$, $x = A$ and $x = B$.

In this section, we shall use the same idea to help us evaluate areas on the plane. First of all, let us consider the following simple example.

EXAMPLE 12.1.1. We wish to find the area of the triangle with vertices $(0, 2)$, $(2, 0)$ and $(2, 4)$. Consider the picture below:



Consider a dissection

$$\Delta: \ 0 = x_0 < x_1 < \ldots < x_n = 2$$

of the interval $[0, 2]$, and suppose that every subinterval $[x_{i-1}, x_i]$ is very short. Suppose that $[x_{i-1}, x_i]$ is the base of the very narrow vertical strip shown in the picture. The heights of the left hand side and right hand side of this vertical strip are respectively

$$(x_{i-1} + 2) - (2 - x_{i-1}) \qquad \text{and} \qquad (x_i + 2) - (2 - x_i).$$

Since $x_i - x_{i-1}$ is very small, the two heights are roughly the same. It follows that the area of this vertical strip is

$$\text{base} \times \text{height} = (x_i - x_{i-1})((\xi_i + 2) - (2 - \xi_i)),$$

where $\xi_i \in [x_{i-1}, x_i]$. If we now consider all such strips, then the total area is the Riemann sum

$$\sum_{i=1}^{n} (x_i - x_{i-1})((\xi_i + 2) - (2 - \xi_i))$$

of the Riemann integral

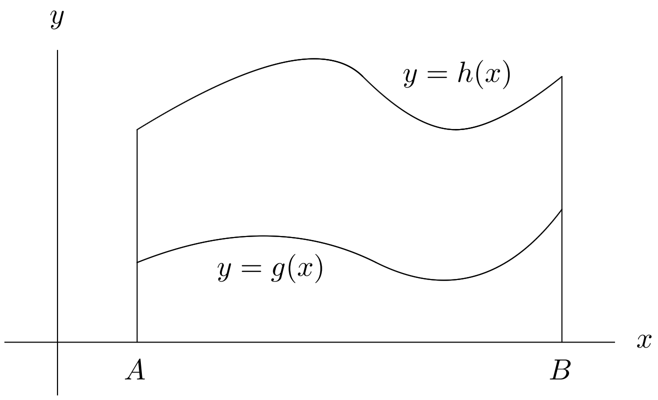
$$\int_0^2 ((x + 2) - (2 - x))\,\mathrm{d}x.$$

It follows that the area of the triangle is

$$\int_0^2 ((x + 2) - (2 - x))\,\mathrm{d}x = \int_0^2 2x\,\mathrm{d}x = 4.$$

Arguing in a similar way, we have the following simple result.

PROPOSITION 12A. *Suppose that the functions $g(x)$ and $h(x)$ are continuous in the closed interval $[A, B]$, where $A, B \in \mathbb{R}$ and $A < B$. Suppose further that $g(x) \le h(x)$ for every $x \in [A, B]$.*



Then the area bounded by the curves $y = g(x)$ and $y = h(x)$ and the lines $x = A$ and $x = B$ is given by

$$\int_A^B (h(x) - g(x))\,\mathrm{d}x.$$

EXAMPLE 12.1.2. Suppose that we wish to find the area α of the triangle with vertices $(0,2)$, $(2,0)$ and $(4,4)$ (the reader should try to draw a picture). We can consider the interval $[0,4]$ and write

$$h(x) = \frac{1}{2}x + 2 \qquad \text{and} \qquad g(x) = \begin{cases} 2-x & \text{if } x \in [0,2], \\ 2x-4 & \text{if } x \in [2,4]. \end{cases}$$

Note that the function $g(x)$ is continuous in the closed interval $[0,4]$. It follows from Proposition 12A that

$$\begin{aligned} \alpha &= \int_0^4 (h(x) - g(x))\,\mathrm{d}x = \int_0^2 (h(x) - g(x))\,\mathrm{d}x + \int_2^4 (h(x) - g(x))\,\mathrm{d}x \\ &= \int_0^2 \left(\frac{1}{2}x + 2 - (2-x)\right) \mathrm{d}x + \int_2^4 \left(\frac{1}{2}x + 2 - (2x-4)\right) \mathrm{d}x \\ &= \frac{3}{2}\int_0^2 x\,\mathrm{d}x + \int_2^4 \left(6 - \frac{3}{2}x\right) \mathrm{d}x = 6. \end{aligned}$$

EXAMPLE 12.1.3. Suppose that we wish to find the area α bounded by the parabola $y^2 = x + 5$ and the line $y = x - 1$ (the reader should try to draw a picture). Note that the parabola intersects the x-axis at the point $(-5,0)$, and that the parabola intersects the line at the points $(4,3)$ and $(-1,-2)$. We can consider the interval $[-5,4]$ and write

$$h(x) = \sqrt{x+5} \qquad \text{and} \qquad g(x) = \begin{cases} -\sqrt{x+5} & \text{if } x \in [-5,-1], \\ x-1 & \text{if } x \in [-1,4]. \end{cases}$$

Note that the function $g(x)$ is continuous in the closed interval $[-5,4]$. It follows from Proposition 12A that

$$\begin{aligned} \alpha &= \int_{-5}^4 (h(x) - g(x))\,\mathrm{d}x = \int_{-5}^{-1} (h(x) - g(x))\,\mathrm{d}x + \int_{-1}^4 (h(x) - g(x))\,\mathrm{d}x \\ &= 2\int_{-5}^{-1} \sqrt{x+5}\,\mathrm{d}x + \int_{-1}^4 (\sqrt{x+5} - x + 1)\,\mathrm{d}x = \frac{125}{6}. \end{aligned}$$

Alternatively, we may interchange the roles of x and y, consider the interval $[-2,3]$ and write

$$H(y) = y + 1 \qquad \text{and} \qquad G(y) = y^2 - 5.$$

It follows from Proposition 12A that

$$\alpha = \int_{-2}^3 (H(y) - G(y))\,\mathrm{d}y = \int_{-2}^3 (6 + y - y^2)\,\mathrm{d}y = \frac{125}{6}.$$

REMARK. Note that in Example 12.1.3, integrating over y proves to be much simpler than integrating over x, as we do not have to break up the range of integration. This is a very important consideration. In choosing which variable to integrate, we must bear in mind two considerations. We want to minimize the number of integrations, and we also want to obtain simple definite integrals. Occasionally a little compromise may be necessary.

12.2. Volumes of Solids

In this section, we first describe a technique for determining the volume of a solid of known cross sectional area.

PROPOSITION 12B. (CAVALIERI'S PRINCIPLE) *Suppose that S is a solid in 3-space, between the planes $x = A$ and $x = B$, where $A < B$. Suppose further that for every $u \in [A, B]$, the cross sectional area of S on the plane $x = u$ (which is perpendicular to the x-axis) is equal to $a(u)$. Then the volume of S is given by*

$$\int_A^B a(x)\,\mathrm{d}x. \tag{1}$$

SKETCH OF PROOF. Consider a dissection

$$\Delta:\ A = x_0 < x_1 < \ldots < x_n = B$$

of the interval $[A, B]$, and suppose that every subinterval $[x_{i-1}, x_i]$ is very short. Then the thin slab that represents the part of the solid S that lies between the planes $x = x_{i-1}$ and $x = x_i$ has volume roughly equal to

$$\text{thickness} \times \text{cross sectional area} = (x_i - x_{i-1})a(\xi_i),$$

where $\xi_i \in [x_{i-1}, x_i]$. If we now consider all such slabs, then the total volume of S is the Riemann sum

$$\sum_{i=1}^{n}(x_i - x_{i-1})a(\xi_i)$$

of the Riemann integral (1). ◯

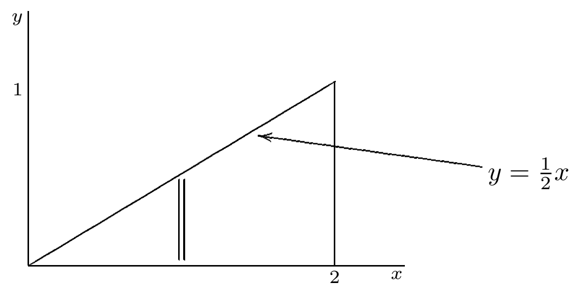
In particular, we are interested in rotating flat areas about a line on the same plane to produce solids of revolution. In this case, the cross sectional area is a circular disc.

PROPOSITION 12C. *Suppose that the function $f(x)$ is continuous in the closed interval $[A, B]$, where $A, B \in \mathbb{R}$ and $A < B$. Suppose further $f(x) \ge 0$ for every $x \in [A, B]$. Then the volume obtained when the area bounded by the curve $y = f(x)$ and the lines $y = 0$, $x = A$ and $x = B$ is rotated about the x-axis is given by*

$$\pi \int_A^B f^2(x)\,\mathrm{d}x.$$

PROOF. Simply note that the area of a circular disc of radius $f(x)$ is $\pi f^2(x)$. ◯

EXAMPLE 12.2.1. Consider the triangle with vertices $(0, 0)$, $(2, 0)$ and $(2, 1)$, as shown in the following diagram:



Here we imagine that the positive z-axis is coming towards us. If we rotate the triangle about the x-axis, then we obtain a solid cone. Consider a dissection

$$\Delta:\ 0 = x_0 < x_1 < \ldots < x_n = 2$$

of the interval $[0,2]$, and suppose that every subinterval $[x_{i-1},x_i]$ is very short. Suppose that $[x_{i-1},x_i]$ is the base of the very narrow vertical strip shown in the picture. The heights of the left hand side and right hand side of this vertical strip are respectively

$$\frac{1}{2}x_{i-1} \qquad \text{and} \qquad \frac{1}{2}x_i.$$

Since $x_i - x_{i-1}$ is very small, the two heights are roughly the same. It follows that when this vertical strip is rotated about the x-axis, we get a very thin circular slab of radius $\frac{1}{2}\xi_i$ and volume

$$\text{thickness} \times \text{area} = (x_i - x_{i-1})\pi\left(\frac{1}{2}\xi_i\right)^2,$$

where $\xi_i \in [x_{i-1},x_i]$. If we now consider all such slabs, then the total volume is the Riemann sum

$$\sum_{i=1}^{n}(x_i - x_{i-1})\pi\left(\frac{1}{2}\xi_i\right)^2$$

of the Riemann integral

$$\int_0^2 \pi\left(\frac{1}{2}x\right)^2 \,\mathrm{d}x.$$

It follows that the volume of the solid cone is

$$\int_0^2 \pi\left(\frac{1}{2}x\right)^2 \,\mathrm{d}x = \frac{\pi}{4}\int_0^2 x^2\,\mathrm{d}x = \frac{2\pi}{3}.$$

Note that Proposition 12C is rather restrictive, in the sense that the x-axis has to feature prominantly as part of the edge of the area in question. The following generalization is much more useful.

PROPOSITION 12D. *Suppose that the functions $g(x)$ and $h(x)$ are continuous in the closed interval $[A,B]$, where $A,B \in \mathbb{R}$ and $A < B$. Suppose further that $h(x) \geq g(x) \geq 0$ for every $x \in [A,B]$. Then the volume obtained when the area bounded by the curves $y = g(x)$ and $y = h(x)$ and the lines $x = A$ and $x = B$ is rotated about the x-axis is given by*

$$\pi\int_A^B (h^2(x) - g^2(x))\,\mathrm{d}x.$$

PROOF. Suppose that $\mathcal{G}$ denotes the area bounded by the curve $y = g(x)$ and the lines $y = 0$, $x = A$ and $x = B$, and that $\mathcal{H}$ denotes the area bounded by the curve $y = h(x)$ and the lines $y = 0$, $x = A$ and $x = B$. Then the area bounded by the curves $y = g(x)$ and $y = h(x)$ and the lines $x = A$ and $x = B$ is obtained by removing $\mathcal{G}$ from $\mathcal{H}$. The volume obtained when this is rotated about the x-axis is therefore the volume obtained when $\mathcal{H}$ is rotated about the x-axis minus the volume obtained when $\mathcal{G}$ is rotated about the x-axis. The result now follows from Proposition 12C. ◯

A natural question to ask at this point is the more general problem of rotating about a line $y = y_0$ rather than the x-axis $y = 0$. We state the following general result.

PROPOSITION 12E. *Suppose that the functions $g(x)$ and $h(x)$ are continuous in the closed interval $[A,B]$, where $A,B \in \mathbb{R}$ and $A < B$. Suppose further that either $h(x) \geq g(x) \geq y_0$ for every $x \in [A,B]$ or $h(x) \leq g(x) \leq y_0$ for every $x \in [A,B]$, where $y_0 \in \mathbb{R}$. Then the volume obtained when the area bounded* *by the curves $y = g(x)$ and $y = h(x)$ and the lines $x = A$ and $x = B$ is rotated about the line $y = y_0$ is given by*

$$\pi \int_A^B ((h(x) - y_0)^2 - (g(x) - y_0)^2)\,\mathrm{d}x.$$

REMARKS. (1) The condition that either $h(x) \geq g(x) \geq y_0$ for every $x \in [A, B]$ or $h(x) \leq g(x) \leq y_0$ for every $x \in [A, B]$ simply means that the curve $y = g(x)$ is between the line $y = y_0$ and the curve $y = h(x)$.

(2) If an area is rotated about the y-axis or a line of the form $x = x_0$, then we have to integrate with respect to y.

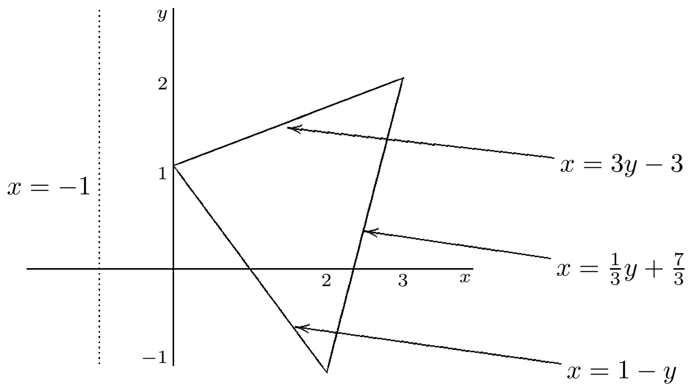
EXAMPLE 12.2.2. Let us try to find the volume of a sphere of radius r. To do this, we shall rotate a half-disc of radius r. More precisely, consider the half-disc on the upper half-plane centred at the origin and of radius r. In other words, we consider the area bounded by the circle $x^2 + y^2 = r^2$, with $y \geq 0$. In the notation of Proposition 12C, we consider the function

$$f(x) = \sqrt{r^2 - x^2}, \quad \text{where } x \in [-r, r].$$

It follows that the volume in question is

$$\pi \int_{-r}^{r} (r^2 - x^2)\,\mathrm{d}x = \frac{4\pi r^3}{3}.$$

EXAMPLE 12.2.3. Consider the triangle with vertices $(0, 1)$, $(2, -1)$ and $(3, 2)$. We wish to find the volume μ when this triangle is rotated about the line $x = -1$. We have the following picture:



We can consider the interval $[-1, 2]$ and write

$$H(y) = \frac{1}{3}y + \frac{7}{3} \qquad \text{and} \qquad G(y) = \begin{cases} 1 - y & \text{if } y \in [-1, 1], \\ 3y - 3 & \text{if } y \in [1, 2]. \end{cases}$$

Note here that the function $G(y)$ is continuous in the closed interval $[-1, 2]$, and that $-1 \leq G(y) \leq H(y)$ in $[-1, 2]$. It follows from Proposition 12E that

$$\begin{aligned} \mu &= \pi \int_{-1}^{2} ((H(y) + 1)^2 - (G(y) + 1)^2)\,\mathrm{d}y \\ &= \pi \int_{-1}^{1} ((H(y) + 1)^2 - (G(y) + 1)^2)\,\mathrm{d}y + \pi \int_{1}^{2} ((H(y) + 1)^2 - (G(y) + 1)^2)\,\mathrm{d}y \\ &= \pi \int_{-1}^{1} \left(\left(\frac{1}{3}y + \frac{10}{3} \right)^2 - (2 - y)^2 \right) \mathrm{d}y + \pi \int_{1}^{2} \left(\left(\frac{1}{3}y + \frac{10}{3} \right)^2 - (3y - 2)^2 \right) \mathrm{d}y = \frac{64\pi}{3}. \end{aligned}$$

12.3. Application to Modelling in Science

If a constant force F is applied to move an object a distance d in the direction of the force, then the work done is given by the product Fd. In general, if the force is a function $F(x)$ in the direction of x, where x denotes the position, then the work done in moving the object from $x = A$ to $x = B$ is given by the definite integral

$$\int_A^B F(x)\,\mathrm{d}x. \tag{2}$$

To see this, consider a dissection

$$\Delta: \ A = x_0 < x_1 < \ldots < x_n = B$$

of the interval $[A, B]$, and suppose that every subinterval $[x_{i-1}, x_i]$ is very short. Then the work done in moving the object between $x = x_{i-1}$ and $x = x_i$ is roughly $F(\xi_i)(x_i - x_{i-1})$, where $\xi_i \in [x_{i-1}, x_i]$. If we consider all such subintervals, then the total work done is given by the Riemann sum

$$\sum_{i=1}^{n}(x_i - x_{i-1})F(\xi_i)$$

of the Riemann integral (2).

Example 12.3.1. An object of mass 200 kilograms is at the end of a chain of length 10 metres, weighing 3 kilogram per metre. The chain with the object is hanging from the top of a building. We wish to find the work done in moving the object and the chain to the top of the building. To do this, let x denote the vertical displacement upwards of the object from its initial position. At this position, the length of the part of the chain which has not reached the top of the building is given by $10-x$, and so the mass of the object and this part of the chain is given by $200+3(10-x)$. Since the force required is to overcome gravity, we have $F(x) = (200 + 3(10 - x))g$. It follows that the total work done is given by

$$\int_0^{10} (200 + 3(10 - x))g\,\mathrm{d}x = \int_0^{10} (230g - 3gx)\,\mathrm{d}x = 2150g.$$

Example 12.3.2. A rectangular tank of height 20 metres and square base of side length 10 metres is half full of water. We wish to find the work done in emptying the tank by pumping the water out over the top, assuming that the density of water is 1000 kilograms per cubic metre. Let x denote the distance from the top. Then the water lies between $x = 10$ and $x = 20$. Consider a dissection

$$\Delta: \ 10 = x_0 < x_1 < \ldots < x_n = 20$$

of the interval $[10, 20]$. Then the part of the water that lies between $x = x_{i-1}$ and $x = x_i$ has volume $100(x_i - x_{i-1})$ and weight $100000(x_i - x_{i-1})$, and this has to be raised a vertical distance of roughly ξ_i, where $\xi_i \in [x_{i-1}, x_i]$. The work done to empty this part of the water is then given by $100000(x_i-x_{i-1})g\xi_i$. Summing the contributions from all the subintervals, we obtain the Riemann sum

$$\sum_{i=1}^{n} 100000(x_i - x_{i-1})g\xi_i$$

of the Riemann integral

$$\int_{10}^{20} 100000gx\,\mathrm{d}x = 15000000g.$$

12.4. Application to Modelling in Economics

Consider the future value F of a payment P made now, or the present value P of a future payment F, where the two quantities P and F are related by an annual interest rate of r, compounded annually over a period of t years. More precisely, we have

$$F = P(1+r)^t \qquad \text{or} \qquad P = \frac{F}{(1+r)^t}.$$

Here we are interested in the corresponding continuous model, known as continuous compounding, where

$$F = P\mathrm{e}^{rt} \qquad \text{or} \qquad P = \frac{F}{\mathrm{e}^{rt}}. \tag{3}$$

In particular, we are interested in a model where the rate of income $R(t)$ at time t, in amounts per year, varies with time t. This is known as an income stream.

Suppose that we wish to estimate the present value of an income stream $R(t)$ in a period from now for a period of M years. Consider a dissection

$$\Delta: \ 0 = t_0 < t_1 < \ldots < t_n = M$$

of the interval $[0, M]$, and suppose that every subinterval $[t_{i-1}, t_i]$ is very short. The income over the period $t \in [t_{i-1}, t_i]$ is roughly equal to $P(\xi_i)(t_i - t_{i-1})$, where $\xi_i \in [t_{i-1}, t_i]$. Using (3), we see that the present value of this income is then roughly $P(\xi_i)(t_i - t_{i-1})\mathrm{e}^{-r\xi_i}$. Summing the contributions from all the subintervals, we conclude that the present value of the entire income stream is given by the Riemann sum

$$\sum_{i=1}^{n} (t_i - t_{i-1}) P(\xi_i) \mathrm{e}^{-r\xi_i}$$

of the Riemann integral

$$V_P = \int_0^M R(t)\mathrm{e}^{-rt}\,\mathrm{d}t.$$

Next, we wish to compute the value at the end of a period of M years from now of an income stream $R(t)$ over this period. Using the same dissection above, we see that the income over the period $t \in [t_{i-1}, t_i]$, in view of (3), has value roughly $P(\xi_i)(t_i - t_{i-1})\mathrm{e}^{r(M\xi_i)}$. Summing the contributions from all the subintervals, we conclude that the value of the entire income stream at the end of the period is given by the Riemann sum

$$\sum_{i=1}^{n} (t_i - t_{i-1}) P(\xi_i) \mathrm{e}^{r(M\xi_i)}$$

of the Riemann integral

$$V_F = \int_0^M R(t)\mathrm{e}^{r(M-t)}\,\mathrm{d}t.$$

REMARK. Note the analogue of (3) for an income stream, given by

$$V_F = \mathrm{e}^{rM} \int_0^M R(t)\mathrm{e}^{-rt}\,\mathrm{d}t = V_P \mathrm{e}^{rM}.$$

Example 12.4.1. A prize winner has the choice of an income of 10000 dollars per year for 20 years, or a lump sum payment of 90000 dollars now, or a lump sum payment of 600000 dollars in 20 years time. The prize winner has to decide which option to exercise, on the assumption that the annual interest rate is 10 per cent compounded continuously. Here we have $R(t) = 10000$ and $r = 0.1$. The income stream satisfies

$$V_P = \int_0^{20} 10000\mathrm{e}^{-0.1t}\,\mathrm{d}t = 100000(1 - \mathrm{e}^{-2}) \approx 86467$$

and

$$V_F = \int_0^{20} 10000\mathrm{e}^{0.1(20-t)}\,\mathrm{d}t \approx 638906.$$

Clearly the prize winner should take the lump sum now.

12.5. Application to Probability Theory

Many of the integrals that arise in this section are not Riemann integrals, as the range of integration is not necessarily finite. In the sense of Riemann, these are called improper integrals. We shall discuss these in Chapter 13. Here we shall only mention some very basic ideas, and we omit any detailed discussion.

The distribution function of a random variable X is the function $F : \mathbb{R} \to [0, 1]$ given by

$$F(x) = \mathrm{Prob}(X \le x),$$

the probability that the value of X does not exceed x. The random variable X is continuous if

$$F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$$

for some integrable function $f : \mathbb{R} \to [0, \infty)$. In this case, the function f is known as the probability density function of X. Note that the numerical value $f(x)$ is not a probability.

Suppose that we wish to find the probability that $X \in (A, B]$. We have

$$\begin{aligned} \mathrm{Prob}(A < X \le B) &= \mathrm{Prob}(X \le B) - \mathrm{Prob}(X \le A) = F(B) - F(A) \\ &= \int_{-\infty}^{B} f(t)\,\mathrm{d}t - \int_{-\infty}^{A} f(t)\,\mathrm{d}t = \int_{A}^{B} f(t)\,\mathrm{d}t. \end{aligned}$$

Since $\mathrm{Prob}(X = A) = 0$, we have

$$\mathrm{Prob}(A \le X \le B) = \int_{A}^{B} f(t)\,\mathrm{d}t.$$

In probability and statistics, we are often interested in the median and the mean of a distribution. The median is a value T such that

$$\mathrm{Prob}(X \le T) = \frac{1}{2}, \qquad \text{or} \qquad \int_{-\infty}^{T} f(t)\,\mathrm{d}t = \frac{1}{2}.$$

The mean is the "average" value of the random variable, and is given by the integral

$$\int_{-\infty}^{\infty} t f(t)\,\mathrm{d}t.$$

12.6. Separable Differential Equations

We consider a differential equation of the type

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y),$$

where $f(x)$ and $g(y)$ are given functions. In some instances, the solution of such an equation reduces to evaluating two indefinite integrals. Indeed, we can separate the variables and obtain

$$\int \frac{\mathrm{d}y}{g(y)} = \int f(x)\,\mathrm{d}x.$$

This enables us to find an expression linking x and y without the derivative.

Example 12.6.1. Suppose that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x-1}{y+1}.$$

Then

$$\int (y+1)\,\mathrm{d}y = \int (x-1)\,\mathrm{d}x, \qquad \text{so that} \qquad \frac{1}{2}y^2 + y = \frac{1}{2}x^2 - x + C.$$

Suppose further that we have the initial condition $y = 1$ when $x = 0$. Then

$$\frac{1}{2}y^2 + y = \frac{1}{2}x^2 - x + \frac{3}{2}.$$

Example 12.6.2. Suppose that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y(x^2-1)}.$$

Then

$$\int y\,\mathrm{d}y = \int \frac{\mathrm{d}x}{x^2-1}, \qquad \text{so that} \qquad y^2 = \log\left|\frac{x-1}{x+1}\right| + C.$$

Suppose further that we have the initial condition $y = 1$ when $x = 0$. Then

$$y^2 = \log\left|\frac{x-1}{x+1}\right| + 1.$$

Example 12.6.3. As an object starts falling under the force of gravity, its downward speed increases. As the speed increases, the air resistence also increases, and this partly balances the pull of gravity. Suppose that the air resistence is proportional to the downward speed, and let us choose the downward direction as the positive direction. Then the downward force is given by $mg - kv$, where m is the mass of the object, g is the acceleration due to gravity, v is the downward speed and k is a positive proportionality constant. Newton's law of motion now gives

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - kv, \qquad \text{or} \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{k}{m}\left(v - \frac{mg}{k}\right).$$

Separating and integrating, we obtain

$$\int \frac{\mathrm{d}v}{v - mg/k} = -\frac{k}{m}\int \mathrm{d}t, \qquad \text{so that} \qquad \log\left|v - \frac{mg}{k}\right| = -\frac{k}{m}t + A.$$

It follows that

$$v - \frac{mg}{k} = C\mathrm{e}^{-kt/m}.$$

At time $t = 0$, the object is stationary. In other words, we have $v = 0$ when $t = 0$. Hence $C = -mg/k$, and so

$$v = \frac{mg}{k}(1 - \mathrm{e}^{-kt/m}).$$

Note that as $t \to +\infty$, we have $v \to mg/k$. This is known as the terminal speed. If we draw a graph of v against t, then the graph is increasing, concave down and has a horizontal asymptote $v = mg/k$.

12.7. Exponential Growth and Decay

If the rate of increase of a certain commodity is directly proportional to the quantity of the commodity, then the commodity is said to have exponential growth. More precisely, we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = kx, \tag{4}$$

where k is a positive proportionality constant. This is usually studied with an initial condition that $x = x_0$ when $t = 0$. Solving the differential equation (4), we obtain the solution

$$x = x_0\mathrm{e}^{kt} \quad \text{for } t \ge 0.$$

This can be adapted to a situation where the commodity may have a rate of growth as well as a rate of decay. For example, if x denotes human population, then we need to look at births as well as deaths. More precisely, we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x - \beta x.$$

If $\alpha > \beta$, then we can study this equation by studying the equation (4) with $k = \alpha - \beta$, and obtain the solution

$$x = x_0\mathrm{e}^{(\alpha-\beta)t} \quad \text{for } t \ge 0.$$

Many populations growing in favourable environment satisfies an equation of the form (4) initially until the huge population size imposes a restriction on further growth and creates a maximum sustainable size. This is usually known as the carrying capacity, and denoted by K. The reality is better described by the logistic model. To understand this, think of the quantity

$$\frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t}$$

as the rate of growth at time t. This growth rate starts at the value k at time $t = 0$, but then declines linearly until it reaches zero when $x = K$. More precisely, we have the differential equation

$$\frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t} = k\left(1 - \frac{x}{K}\right). \tag{5}$$

Note that this equation is restricted to $0 \le x \le K$. It is a simple exercise to show that if $x = x_0 < K$ when $t = 0$, then the solution of the equation (5) is given by

$$x = \frac{Kx_0}{x_0 + (K - x_0)\mathrm{e}^{-kt}} \quad \text{for } t \ge 0,$$

with $x < K$ for $t \geq 0$, and

$$\lim_{t \to +\infty} x = K.$$

Note also from (5) that $\mathrm{d}x/\mathrm{d}t > 0$.

If the rate of decrease of a certain commodity is directly proportional to the quantity of the commodity, then the commodity is said to have exponential decay. More precisely, we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -kx, \tag{6}$$

where k is a positive proportionality constant. This is usually studied with an initial condition that $x = x_0$ when $t = 0$. Solving the differential equation (6), we obtain the solution

$$x = x_0 \mathrm{e}^{-kt} \quad \text{for } t \geq 0.$$

Problems for Chapter 12

1. Find the area enclosed by the curves $y = x^2$ and $y = x^4$.

2. Find the area enclosed by the four lines $x + y = 1$, $x + y = 5$, $x - 3y = 1$ and $x - 3y = -3$.

3. Find the area bounded by the curve $x = 4y - 4y^2$ and the lines $x - y = 3$, $y = 0$ and $y = 1$.

4. Let R be the region in the first quadrant bounded by the curve $y = \sin^{-1} x$, the x-axis and the line $x = 1/2$. Give a sketch of the region R and determine its area.

5. Suppose that α is a positive real number and $n > 1$ is an integer.
   a) Find the area S_n bounded by the curve $y = \dfrac{2x}{(x^2+1)^\alpha}$ and the x-axis between the lines $x = 1$ and $x = n$.
   b) Find all values of α for which the limit of S_n is finite as $n \to +\infty$.

6. Use integration to show that the volume of a sphere of radius R is $\frac{4}{3}\pi R^3$.

7. For each of the lines below, find the volume obtained when the area bounded by the parabolas $y = 1 - x^2$ and $y = 3 - 3x^2$ is rotated about the line:
   a) $y = 0$ b) $y = -2$ c) $y = 4$

8. a) Sketch the curves $y = x$ and $y = \mathrm{e}^x$ on the same coordinate plane.
   b) Find the area bounded by the two curves in part (a), the y-axis and the line $x = 1$.
   c) Find the volume created when the area in part (b) is rotated about the x-axis.
   d) Find the volume created when the area in part (b) is rotated about the y-axis.

9. Let S be the region bounded by the curve $y = \cos 2x$ for $0 \le x \le \pi/4$, the x-axis and the y-axis. Determine the volume generated when the region S is rotated about the line $x = -1$.

10. A group of workers at the top of a building of height 100 metres need to lift a weight of 500 kilograms from the ground to a height of 30 metres using a chain weighing 2 kilograms per metre. Find the work done.

11. A rectangular tank, of length 20 metres, width 10 metres and height 15 metres, is two thirds full of water. Find the work done in emptying the tank by pumping the water out over the top, assuming that the density of water is 1000 kilograms per cubic metre.

12. A physicist wishes to build a sandcastle in the shape of a cone, of height 50 metres and base radius 10 metres. Find the work done, assuming that the density of sand is 300 kilograms per cubic metre, and assuming that there is no collapse of sand in the process.

13. A cylindrical barrel of radius 2 metres and height 4 metres is three quarters full of muddy water. The density of the muddy water at a depth of x metres from the surface is given by $\delta(x) = 1000+3x$, in kilograms per cubic metre. Find the work done in emptying the water by pumping it out over the top of the barrel.

14. Assume that annual interest rate is at 10 per cent compounded continuously.
    a) At what constant rate must money be deposited so that the value of the deposit is worth 100000 dollars at the end of 10 years?
    b) Suppose that money is deposited at the rate of $R(t) = 7000(1 + 0.2t)$. What is the value of the deposit at the end of 10 years?

15. A football player is offered a new 5 year contract. He has a choice of a salary rate of 2 million dollars per year, or a lump sum of 2 million dollars at the start of the contract together with a salary rate of 1 million dollars per year. His manager advises him that any money can be invested with an annual return of 50 per cent compounded continuously. What choice should the player make?

16. Water leaks from a container at a rate proportional to the square root of the depth of the water at the time. Suppose that the water level starts at 100 centimetres and drops to 95 centimetres in an hour. Determine how long it will take for all the water to leak out.

17. A spherical block of ice starts with radius 1 metre. Suppose that the ice melts at a rate proportional to the surface area. After one hour, the radius is reduced to 50 centimetres. How long does it take the radius to reach 20 centimetres?

18. Find a real valued function $f(x)$, defined for all $x \geq 0$, such that for every $x_0 \geq 0$, the tangent to the curve $y = f(x)$ at the point $(x_0, f(x_0))$ intersects the x-axis at the point $x = -x_0$.

19. The rate at which material is forgotten is proportional to the difference between the amount of material currently remembered and some small positive constant c. Here if $x(t)$ is the proportion of material remembered at time t, then $x(t) \geq c$ for every $t \geq 0$. Write down a differential equation relating x and t, solve the differential equation, and comment on the solution.
[Remark: This is of course in the hope that you have not forgotten how to solve simple differential equations.]

20. An object of mass m is fired vertically upwards with initial velocity v_0, with the intention that it will escape the pull of gravity and escape from the earth. Suppose that the gravitational force F on the object at an altitude of h above the surface of the earth is given by

$$F = \frac{mgR^2}{(R+h)^2},$$

where R is the radius of the earth.

a) Let v denote the upward velocity of the object at time t. Use Newton's law of motion to show that

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{gR^2}{(R+h)^2}.$$

b) Use the Chain rule $\dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{\mathrm{d}v}{\mathrm{d}h}\dfrac{\mathrm{d}h}{\mathrm{d}t}$ to rewrite the differential equation as one with h as independent variable.

c) Solve the new differential equation in part (b).

d) Find the smallest value of v_0 such that v is never equal to zero at any time t.

21. Suppose that a population grows in accordance with the equation (4).

a) Suppose further that the population doubles over the first 5 hours. Show that the population doubles over any arbitrary continuous period of 5 hours.

b) What conclusion can you make if the population doubles over the first T hours? Judtify your assertion.

22. Study the logistic equation in the case when $x_0 > K$.

23. Consider the logistic equation. Suppose that x_0, x_1 and x_2 are the values of x at time $t = 0$, $t = T$ and $t = 2T$ respectively.

a) Show that

$$\frac{x_1}{x_0}\mathrm{e}^{-kT} = \frac{K - x_1}{K - x_0} \qquad \text{and} \qquad \frac{x_2}{x_0}\mathrm{e}^{-2kT} = \frac{K - x_2}{K - x_0}.$$

b) Find an expression for K in terms of x_0, x_1 and x_2.

24. Suppose that a population decays in accordance with the equation (6). The time T taken for the population to decay from its initial quantity x_0 to $\frac{1}{2}x_0$ is known as the half-life of the population.
   a) Find the value of T in terms of the proportionality constant k.
   b) Show that the population halves over any period of length T.

25. Intravenous infusion is often modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -kx + c,$$

   where x denotes the concentration in the blood at time t, k is a positive proportionality constant, and c is another positive constant that represents the rate of drug administration.
   a) Find the constant solution of the differential equation.
   [REMARK: This is known as the equilibrium solution.]
   b) Suppose that $x = x_0$ at time $t = 0$. Find the concentration x at time t. What happens when $t \to +\infty$? Compare this to the solution in part (a).
   c) Sketch the graph of a typical solution.