FIRST YEAR CALCULUS

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Chapter 14

ORDINARY DIFFERENTIAL EQUATIONS

14.1. Introduction

Any equation containing differential coefficients is called a differential equation. Ordinary differential equations are those that involve only one independent variable and therefore only ordinary differential coefficients.

Usually the independent variable is denoted by x and the dependent variable is denoted by y, and we think of y as a function of x. An ordinary differential equation is therefore any function of x, y and the derivatives of y such that

$$F\left(x,y,\frac{\mathrm{d}y}{\mathrm{d}x},\frac{\mathrm{d}^2y}{\mathrm{d}x^2},\ldots\right)=0.$$

EXAMPLE 14.1.1. The ordinary differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=5y$$

is of order 1 and degree 1.

EXAMPLE 14.1.2. The ordinary differential equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^4+y^2=x$$

is of order 1 and degree 4.

Example 14.1.3. The ordinary differential equation

$$\frac{d^3y}{dx^3}+5\frac{d^2y}{dx^2}+4\frac{dy}{dx}+y=\cos x$$

is of order 3 and degree 1.

Example 14.1.4. The ordinary differential equation

$$\frac{d^2y}{dx^2}+5\left\{\left(\frac{dy}{dx}\right)^2+y\right\}^{1/3}=0$$

is of order 2 and degree 3.

We now define the order and degree of a differential equation.

Definition. The order of an ordinary differential equation is the order of the highest differential coefficient contained in it. The degree of an ordinary differential equation is the power to which the highest order differential coefficient is raised when the equation is rationalized; in other words, when fractional powers are removed.

Example 14.1.5. In Example 14.1.4, the ordinary differential equation can be written in rationalized form as

$$\left(\frac{d^2y}{dx^2}\right)^3+125\left\{\left(\frac{dy}{dx}\right)^2+y\right\}=0.$$

Definition. An ordinary differential equation of order n is said to be linear if it is linear in the dependent variable y and linear in each of the derivatives

$$\frac{dy}{dx},\frac{d^2y}{dx^2},\ldots,\frac{d^ny}{dx^n}.$$

Otherwise, the ordinary differential equation is said to be non-linear.

Example 14.1.6. The ordinary differential equations in Examples 14.1.1 and 14.1.3 above are linear, while those in Examples 14.1.2 and 14.1.4 are non-linear.

Example 14.1.7. The ordinary differential equation

$$\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)=5y$$

is non-linear and of order 2 and degree 1.

Non-linear ordinary differential equations are usually very difficult, with standard techniques only for very few cases. We shall discuss a few such techniques as applied to first order ordinary differential equations.

14.2. How Ordinary Differential Equations Arise

We shall first of all consider a few examples. Do not worry about the details.

EXAMPLE 14.2.1. Consider the equation $y^2 = 4A(x+A)$, where A is a constant. Differentiating once, we obtain

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4A, \qquad \text{so that} \qquad A = \frac{y}{2}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Substituting A into the original equation and simplifying, we obtain the first order non-linear equation

$$y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2x\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0.$$

EXAMPLE 14.2.2. Suppose that $y = (A+Bx)\mathrm{e}^{3x}$, where A and B are constants. If we differentiate twice, then we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = B\mathrm{e}^{3x} + 3(A+Bx)\mathrm{e}^{3x} \qquad \text{and} \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6B\mathrm{e}^{3x} + 9(A+Bx)\mathrm{e}^{3x}.$$

Writing

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} \qquad \text{and} \qquad y'' = \frac{\mathrm{d}^2y}{\mathrm{d}x^2},$$

the three equations can now be described in matrix form as

$$\begin{pmatrix} \mathrm{e}^{3x} & x\mathrm{e}^{3x} & y \\ 3\mathrm{e}^{3x} & (3x+1)\mathrm{e}^{3x} & y' \\ 9\mathrm{e}^{3x} & (9x+6)\mathrm{e}^{3x} & y'' \end{pmatrix} \begin{pmatrix} A \\ B \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since e^{3x} is non-zero, we must therefore have

$$\det \begin{pmatrix} 1 & x & y \\ 3 & 3x+1 & y' \\ 9 & 9x+6 & y'' \end{pmatrix} = 0.$$

Evaluating the determinant gives rise to the second order linear equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 0.$$

EXAMPLE 14.2.3. Suppose that $y = A\mathrm{e}^{-x} + B\mathrm{e}^{-2x} + C\mathrm{e}^{3x}$, where A, B and C are constants. If we differentiate three times, then we obtain $y' = -A\mathrm{e}^{-x} - 2B\mathrm{e}^{-2x} + 3C\mathrm{e}^{3x}$, $y'' = A\mathrm{e}^{-x} + 4B\mathrm{e}^{-2x} + 9C\mathrm{e}^{3x}$ and $y''' = -A\mathrm{e}^{-x} - 8B\mathrm{e}^{-2x} + 27C\mathrm{e}^{3x}$. The four equations can now be described in matrix form as

$$\begin{pmatrix} \mathrm{e}^{-x} & \mathrm{e}^{-2x} & \mathrm{e}^{3x} & y \\ -\mathrm{e}^{-x} & -2\mathrm{e}^{-2x} & 3\mathrm{e}^{3x} & y' \\ \mathrm{e}^{-x} & 4\mathrm{e}^{-2x} & 9\mathrm{e}^{3x} & y'' \\ -\mathrm{e}^{-x} & -8\mathrm{e}^{-2x} & 27\mathrm{e}^{3x} & y''' \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since e^{-x}, e^{-2x} and e^{3x} are all non-zero, we must therefore have

$$\det \begin{pmatrix} 1 & 1 & 1 & y \\ -1 & -2 & 3 & y' \\ 1 & 4 & 9 & y'' \\ -1 & -8 & 27 & y''' \end{pmatrix} = 0.$$

Evaluating the determinant gives rise to the second order linear equation

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 7\frac{\mathrm{d}y}{\mathrm{d}x} - 6y = 0.$$

Note that in these three examples, the expression of y as a function of x contains respectively one, two and three constants. By differentiating this expression respectively once, twice and three times, we are in a position to eliminate these constants.

In general, if the expression of y as a function of x contains n arbitrary constants, then differentiating n times, we obtain n further equations. We now have $(n+1)$ equations containing these n constants, and we expect to be able (at least theoretically) to eliminate these constants. After eliminating these constants, we expect to end up with an ordinary differential equation of order n.

The above approach can sometimes be varied, as illustrated in the next example.

EXAMPLE 14.2.4. The general circle on a plane is given by the equation $(x-A)^2 + (y-B)^2 = R^2$, where A, B and R are constants. If we differentiate three times instead of twice, we obtain the equations $(x-A)+(y-B)y' = 0$, $1+(y-B)y''+(y')^2 = 0$ and $(y-B)y'''+3y'y'' = 0$. These last three equations can be written in matrix form as

$$\begin{pmatrix} 1 & y' & 0 \\ 0 & y'' & 1+(y')^2 \\ 0 & y''' & 3y'y'' \end{pmatrix} \begin{pmatrix} x-A \\ y-B \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Evaluating the determinant gives rise to the equation

$$3\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)^2 = \frac{\mathrm{d}^3y}{\mathrm{d}x^3}\left(1+\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)^2\right).$$

This looks like a third order equation. However, if we write $u = \dfrac{\mathrm{d}y}{\mathrm{d}x}$, then the equation becomes

$$3u\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 = \frac{\mathrm{d}^2u}{\mathrm{d}x^2}\left(1+\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\right),$$

a second order equation.

If we reverse the argument, it is reasonable to define the general solution of an ordinary differential equation of order n as that solution containing n arbitrary constants. This is, however, not entirely satisfactory. Instead, the following is true: Any solution of an ordinary differential equation of order n containing fewer than n arbitrary constants cannot be the general solution.

In many physical problems, the solution of a differential equation has to satisfy certain specified conditions. These are called initial or boundary conditions, and determine the values of the arbitrary constants in the solution.

EXAMPLE 14.2.5. Consider again the ordinary differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 0.$$

The general solution is $y = (A+Bx)\mathrm{e}^{3x}$. Suppose that we have the initial conditions $y = 1$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 6$ when $x = 0$. Then we must have $y = (1+3x)\mathrm{e}^{3x}$.

14.3. Some Modelling Problems

In this section, we give a few simple examples from physics where ordinary differential equations are used to describe the physical phenomena. For the first few examples in mechanics, it is convenient to use t to denote the independent variable representing time, and to use x as the dependent variable representing displacement.

Example 14.3.1. Consider a body falling near the surface of the earth. If we neglect air resistence, then the body is subject to a constant force $F = -mg$, where m denotes the mass of the body and g denotes gravity. This force is negative if we adopt the convention that the positive direction is upwards. Using Newton's law, the equation of motion is given by

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -mg, \qquad \text{or simply} \qquad \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -g.$$

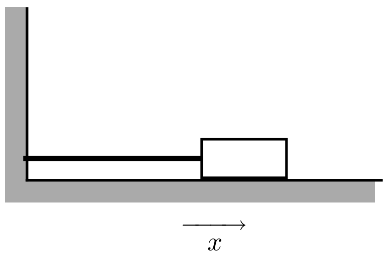
Example 14.3.2. Suppose that in the last example, we no longer neglect air resistence, but assume instead a frictional force proportional to the speed of the body. Then the body is subject to a force

$$F = -mg - b\frac{\mathrm{d}x}{\mathrm{d}t},$$

where $b > 0$ is a fixed proportionality constant. Using Newton's law, the equation of motion is now given by

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -mg - b\frac{\mathrm{d}x}{\mathrm{d}t}, \qquad \text{or} \qquad m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + mg = 0.$$

Example 14.3.3. Consider a body of mass m fastened to a spring whose constant is k. If we measure the position x of the body from the relaxed position of the spring, with the convention that the positive direction is to the right, as shown in the picture below, then the spring exerts a restoring force $F = -kx$.



If we neglect friction and assume that there are no other forces, then using Newton's law, the equation of motion is given by

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx, \qquad \text{or} \qquad m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + kx = 0.$$

Example 14.3.4. Suppose that in the last example, we no longer neglect friction, but assume instead a frictional force proportional to the speed of the body. Then the body is subject to a force

$$F = -kx - b\frac{\mathrm{d}x}{\mathrm{d}t}.$$

Using Newton's law, the equation of motion is now given by

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx - b\frac{\mathrm{d}x}{\mathrm{d}t}, \qquad \text{or} \qquad m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0.$$

Example 14.3.5. Suppose that in the last example, the body is subject to an additional impressed force $F(t)$. Then it is subject to a total force

$$F = F(t) - kx - b\frac{\mathrm{d}x}{\mathrm{d}t}.$$

Using Newton's law, the equation of motion is now given by

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = F(t) - kx - b\frac{\mathrm{d}x}{\mathrm{d}t}, \qquad \text{or} \qquad m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F(t).$$

A problem in electrical circuits is analogous to our last example.

Example 14.3.6. Consider an electric circuit containing in series a resistance R, a capacitance C, an inductance L and a source of electromotive force E. Suppose that the current flowing around the circuit at time t is given by $I(t)$, and that the charge on the capacitor is $q(t)$. Then

$$I = \frac{\mathrm{d}q}{\mathrm{d}t}. \tag{1}$$

The voltage across the resistor is RI, the voltage across the capacitor is q/C, and the voltage across the inductor is

$$L\frac{\mathrm{d}I}{\mathrm{d}t}.$$

Then at any time t, we have

$$L\frac{\mathrm{d}I}{\mathrm{d}t} + RI + \frac{q}{C} = E.$$

If we now differentiate with respect to t and use the relation (1), then we have

$$L\frac{\mathrm{d}^2I}{\mathrm{d}t^2} + R\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{I}{C} = \frac{\mathrm{d}E}{\mathrm{d}t}.$$

We shall discuss the solutions of some of these examples in Chapters 15 and 16.

Problems for Chapter 14

1. Suppose that $y = A\cos x + B\sin x + Ce^x$, where A, B and C are constants. Show that y is a solution of an ordinary differential equation.

2. Suppose that $y = (A + Bx + Cx^2)e^{-x}$, where A, B and C are constants. Show that y is a solution of an ordinary differential equation.

3. For each of the following, find a differential equation of which the given expression is the general solution, with A and B being arbitrary constants:
   a) $y = e^{-kx}(A\cos nx + B\sin nx)$
   b) $y = Ae^{-x} + Bx$
   c) $y = (x + A)\sin x$