FIRST YEAR CALCULUS

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Chapter 15

FIRST ORDER
ORDINARY DIFFERENTIAL EQUATIONS

15.1. Introduction

First order ordinary differential equations are of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = F(x,y), \tag{1}$$

where $F(x,y)$ is a given function. Although this equation looks simple, solutions can usually only be found easily when the function $F(x,y)$ has particularly simple forms.

15.2. Separable Variable Type

This is an ordinary differential equation of the type (1), where $F(x,y)$ is of the form $f(x)g(y)$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y).$$

We can therefore separate the variables and obtain

$$\int \frac{\mathrm{d}y}{g(y)} = \int f(x)\,\mathrm{d}x.$$

This gives an expression of y in terms of x.

EXAMPLE 15.2.1. Suppose that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x-1}{y+1}.$$

Then

$$\int (y+1)\,\mathrm{d}y = \int (x-1)\,\mathrm{d}x, \qquad \text{so that} \qquad \frac{1}{2}y^2 + y = \frac{1}{2}x^2 - x + C.$$

Suppose further that we have the initial condition $y = 1$ when $x = 0$. Then

$$\frac{1}{2}y^2 + y = \frac{1}{2}x^2 - x + \frac{3}{2}.$$

EXAMPLE 15.2.2. Suppose that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y(x^2-1)}.$$

Then

$$\int y\,\mathrm{d}y = \int \frac{\mathrm{d}x}{x^2-1}, \qquad \text{so that} \qquad y^2 = \log\left|\frac{x-1}{x+1}\right| + C.$$

Suppose further that we have the initial condition $y = 1$ when $x = 0$. Then

$$y^2 = \log\left|\frac{x-1}{x+1}\right| + 1.$$

15.3. The Homogeneous Equation

This is an ordinary differential equation of the type (1), where $F(x,y)$ is of the form $f(x,y)/g(x,y)$, where $f(x,y)$ and $g(x,y)$ are homogeneous functions of x and y of some degree k, say; in other words, $f(ax,ay) = a^k f(x,y)$ and $g(ax,ay) = a^k g(x,y)$ for every $a \in \mathbb{R}$. Then clearly

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f(x,y)}{g(x,y)} = \frac{x^k f(1,y/x)}{x^k g(1,y/x)} = \frac{f(1,y/x)}{g(1,y/x)} = \phi\left(\frac{y}{x}\right), \tag{2}$$

for some function ϕ. Using the substitution

$$y = vx, \tag{3}$$

this equation can be reduced to an equation of separable variable type. Indeed, differentiating (3), we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}. \tag{4}$$

Combining (2)–(4), we have

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \phi(v),$$

so that on separating the variables, we have

$$\int \frac{\mathrm{d}v}{\phi(v) - v} = \int \frac{\mathrm{d}x}{x}.$$

We can therefore express v (and hence y) in terms of x.

EXAMPLE 15.3.1. Suppose that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + y^2}{2xy}.$$

Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 + (y/x)^2}{2(y/x)}.$$

Using the substitution (3), we obtain

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1 + v^2}{2v},$$

so that on separating the variables, we have

$$\int \frac{2v\,\mathrm{d}v}{1 - v^2} = \int \frac{\mathrm{d}x}{x}.$$

Integrating, we obtain $-\log(1 - v^2) = \log x + A$, so that

$$\frac{x^2}{x^2 - y^2} = \frac{1}{1 - v^2} = \mathrm{e}^A x.$$

This gives $x^2 - y^2 = Cx$ for some constant C.

EXAMPLE 15.3.2. Suppose that

$$3xy^2\frac{\mathrm{d}y}{\mathrm{d}x} = x^3 + y^3.$$

Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 + (y/x)^3}{3(y/x)^2}.$$

Using the substitution (3), we obtain

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1 + v^3}{3v^2},$$

so that on separating the variables, we have

$$\int \frac{3v^2\,\mathrm{d}v}{1 - 2v^3} = \int \frac{\mathrm{d}x}{x}.$$

Integrating, we obtain $-\log(1 - 2v^3) = 2\log x + A$, so that

$$\frac{x^3}{x^3 - 2y^3} = \frac{1}{1 - 2v^3} = \mathrm{e}^A x^2.$$

This gives $x^3 - 2y^3 = Cx$ for some constant C.

Some non-homogeneous equations can be transformed into homogeneous form by a simple change of variables. Consider the following simple example.

EXAMPLE 15.3.3. Suppose that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x+y+3}{x+2y+9}.$$

Then writing $x = X + x_0$ and $y = Y + y_0$, where X and Y are new variables and x_0 and y_0 are constants, we obtain

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2X+Y+2x_0+y_0+3}{X+2Y+x_0+2y_0+9}. \tag{5}$$

If we choose $x_0 = 1$ and $y_0 = -5$, then the equation (5) reduces to

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{2X+Y}{X+2Y},$$

which is now a homogeneous equation. Using the substitution $Y = vX$, we can show that this equation has solution $(X-Y)^3(X+Y) = C$ for some constant C, so that $(x-y-6)^3(x+y+4) = C$.

Indeed, if

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ax+by+c}{fx+gy+h}, \tag{6}$$

then substituting

$$x = X + x_0 \qquad \text{and} \qquad y = Y + y_0, \tag{7}$$

where X and Y are new variables and x_0 and y_0 are constants, we obtain

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{aX+bY+ax_0+by_0+c}{fX+gY+fx_0+gy_0+h}, \tag{8}$$

which is homogeneous provided that

$$ax_0+by_0+c = 0 \qquad \text{and} \qquad fx_0+gy_0+h = 0. \tag{9}$$

We therefore choose x_0 and y_0 so that (9) is satisfied. Then (8) becomes

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{aX+bY}{fX+gY},$$

and this can be solved by the substitution $v = Y/X$. We then obtain Y in terms of X, so that in view of (7), we obtain y in terms of x.

Note that the technique above depends on the existence of constants x_0 and y_0 which satisfy (9). We can rephrase the problem as follows: Consider two lines on the plane given by the equations

$$ax+by+c = 0 \qquad \text{and} \qquad fx+gy+h = 0.$$

As long as these two lines are not parallel, then they intersect at precisely one point. This point is given by (x_0, y_0).

However, these two lines may be different and parallel, so that there will be no such intersection points. In this case, the technique above breaks down. In this case, we use a different technique illustrated by our next example.

EXAMPLE 15.3.4. Suppose that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y+3}{x+y+4}.$$

Write $u = x + y$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 1 + \frac{\mathrm{d}y}{\mathrm{d}x}, \qquad \text{so that} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 1 + \frac{u+3}{u+4}.$$

Integrating, we obtain $2u + 7 = \mathrm{e}^{2(2x-u+C)}$. Hence $2x + 2y + 7 = \mathrm{e}^{2(x-y+C)}$.

Indeed, if

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ax+by+c}{fx+gy+h}, \qquad \text{where} \qquad ag = bf,$$

then we can write $fx + gy + h = k(ax + by) + h$ for some real number k. Now write $u = ax + by$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = a + b\frac{\mathrm{d}y}{\mathrm{d}x}, \qquad \text{so that} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = a + b\frac{u+c}{ku+h}.$$

The equation is now reduced to one of separable variable type.

15.4. The Linear Equation

This is an ordinary differential equation of the type (1), where $F(x, y)$ is of the form $Q(x) - P(x)y$, where $P(x)$ and $Q(x)$ are two given functions. We therefore consider equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x). \tag{10}$$

Note that this ordinary differential equation is linear, and is called the general linear first order ordinary differential equation.

Equation (10) may be solved with the help of an integrating factor $\mu(x)$. Multiplying (10) by such an integrating factor, we obtain

$$\mu(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \mu(x)P(x)y = \mu(x)Q(x). \tag{11}$$

The integrating factor $\mu(x)$ is chosen in order to make the left hand side of (11) equal to $\dfrac{\mathrm{d}}{\mathrm{d}x}(\mu(x)y)$. We must therefore have

$$\mu(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \mu(x)P(x)y = \frac{\mathrm{d}}{\mathrm{d}x}(\mu(x)y) = \mu(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{\mathrm{d}}{\mathrm{d}x}\mu(x)\right)y, \tag{12}$$

and this can be achieved if we take $\mu(x)$ to satisfy

$$\mu(x)P(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mu(x). \tag{13}$$

Integrating (13), we have

$$\int \frac{\mathrm{d}\mu}{\mu} = \int P(x)\,\mathrm{d}x,$$

so that the choice

$$\mu(x) = \mathrm{e}^{\int P(x)\,\mathrm{d}x} \tag{14}$$

will be suitable. The equation (11) then becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu(x)y) = \mu(x)Q(x),$$

giving the solution

$$\mu(x)y = \int \mu(x)Q(x)\,\mathrm{d}x. \tag{15}$$

This gives y in terms of x. Note also that since $P(x)$ is given, an integrating factor $\mu(x)$ is easily determined by (14).

EXAMPLE 15.4.1. Suppose that

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{3}{x}y = x^2.$$

Here $P(x) = 3/x$ and $Q(x) = x^2$. Also $\int P(x)\,\mathrm{d}x = 3\log x$, so it follows from (14) that an integrating factor is

$$\mu(x) = \mathrm{e}^{3\log x} = x^3.$$

It follows from (15) that the solution is given by

$$x^3y = \int x^3x^2\,\mathrm{d}x = \int x^5\,\mathrm{d}x = \frac{x^6}{6} + C$$

for some constant C. Hence

$$y = \frac{x^3}{6} + \frac{C}{x^3}.$$

EXAMPLE 15.4.2. Suppose that

$$(x^2+3x+2)\frac{\mathrm{d}y}{\mathrm{d}x} + xy = x(x+1).$$

Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x}{x^2+3x+2}y = \frac{x(x+1)}{x^2+3x+2}.$$

Here

$$P(x) = \frac{x}{x^2+3x+2} \qquad \text{and} \qquad Q(x) = \frac{x(x+1)}{x^2+3x+2} = \frac{x}{x+2}.$$

Also

$$\int P(x)\,\mathrm{d}x = \int \frac{x}{x^2+3x+2}\,\mathrm{d}x = \int \left(\frac{2}{x+2} - \frac{1}{x+1}\right)\mathrm{d}x = \log\left(\frac{(x+2)^2}{x+1}\right),$$

so it follows from (14) that an integrating factor is

$$\mu(x) = \frac{(x+2)^2}{x+1}.$$

It follows from (15) that the solution is given by

$$\begin{aligned}\frac{(x+2)^2}{x+1}y &= \int \frac{(x+2)^2}{(x+1)}\frac{x}{(x+2)}\,\mathrm{d}x = \int \frac{x(x+2)}{x+1}\,\mathrm{d}x \\ &= \int \left(x+1-\frac{1}{x+1}\right)\mathrm{d}x = \frac{x^2}{2}+x-\log(x+1)+C\end{aligned}$$

for some constant C. Hence

$$y = \frac{x(x+1)}{2(x+2)} - \frac{(x+1)}{(x+2)^2}\log(x+1) + C\frac{(x+1)}{(x+2)^2}.$$

15.5. Application to a Problem in Physics

In this section, we study a differential equation first discussed in Section 14.3. For this example in mechanics, it is convenient to use t to denote the independent variable representing time, and to use x as the dependent variable representing displacement.

EXAMPLE 15.5.1. In Examples 14.3.1 and 14.3.2, we consider a body falling near the surface of the earth subject to a constant force $F = -mg$, where m denotes the mass of the body and g denotes gravity, and a frictional force proportional to the speed of the body. Recall that the equation of motion is given by

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + mg = 0,$$

where $b > 0$ is a fixed proportionality constant. This equation can be treated as a first order differential equation in the variable $v = \mathrm{d}x/\mathrm{d}t$, and written in the form

$$m\frac{\mathrm{d}v}{\mathrm{d}t} + bv + mg = 0, \qquad \text{or} \qquad \frac{\mathrm{d}v}{\mathrm{d}t} + \frac{b}{m}v = -g.$$

This is therefore a standard first order linear equation or the form

$$\frac{\mathrm{d}v}{\mathrm{d}t} + P(t)v = Q(t),$$

where $P(t) = b/m$ and $Q(t) = -g$. Note that

$$\int P(t)\,\mathrm{d}t = \frac{bt}{m},$$

so that the integrating factor is $\mu(t) = \mathrm{e}^{bt/m}$, and the equation can be rewritten in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{bt/m}v) = -g\mathrm{e}^{bt/m},$$

so that

$$\mathrm{e}^{bt/m}v = -g\int \mathrm{e}^{bt/m}\,\mathrm{d}t = C - \frac{mg}{b}\mathrm{e}^{bt/m}, \qquad \text{and so} \qquad v = C\mathrm{e}^{-bt/m} - \frac{mg}{b}.$$

Here C is an absolute constant. It follows that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = C\mathrm{e}^{-bt/m} - \frac{mg}{b},$$

so that

$$x = \int \left(C\mathrm{e}^{-bt/m} - \frac{mg}{b} \right) \mathrm{d}t = C_1\mathrm{e}^{-bt/m} - \frac{mg}{b}t + C_2,$$

where C_1 and C_2 are absolute constants, to be determined by initial conditions.

Problems for Chapter 15

1. Find the general solution of the differential equation

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x}+xy-y^2=0$$

by using the substitution $y=ux$.

2. For each of the following differential equations, find its general solution:

a) $(1+x^2)\dfrac{\mathrm{d}y}{\mathrm{d}x}+4xy=0$

b) $(x^2+1)\dfrac{\mathrm{d}y}{\mathrm{d}x}+xy=x$

c) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2x+2y-2}{3x+y-5}$

d) $xy\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{x^2+1}{y^2-1}$

e) $\dfrac{\mathrm{d}y}{\mathrm{d}x}-y\cot x=\sin x$

f) $(x+1)\dfrac{\mathrm{d}y}{\mathrm{d}x}-3y=(x+1)^5$

g) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{3x+y+6}{6x+2y+9}$

h) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{x^2+xy+y^2}{x^2}$

i) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{y}{x}+\dfrac{y^2}{x^2}$

j) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{x+y-1}{x+y+1}$

k) $\dfrac{\mathrm{d}y}{\mathrm{d}x}+2xy=2\mathrm{e}^{-x^2}$

l) $x\dfrac{\mathrm{d}y}{\mathrm{d}x}-y=xy$

m) $(1+x^2)\dfrac{\mathrm{d}y}{\mathrm{d}x}=y^2$

n) $x^2\dfrac{\mathrm{d}y}{\mathrm{d}x}=x^2-xy+y^2$

o) $\dfrac{\mathrm{d}y}{\mathrm{d}x}+\dfrac{y}{x}=\sin x$

p) $\dfrac{\mathrm{d}y}{\mathrm{d}x}+y\tan x=x\sin 2x$

q) $x\dfrac{\mathrm{d}y}{\mathrm{d}x}+2y=\mathrm{e}^x$

r) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^{x+y}$

3. Solve each of the following differential equation with the given initial condition:

a) $\dfrac{\mathrm{d}y}{\mathrm{d}x}+2y\tan x=\sin x$, with $y(\pi/3)=0$

b) $x\dfrac{\mathrm{d}y}{\mathrm{d}x}=x+y$, with $y(1)=1$

c) $(1-x^2)\dfrac{\mathrm{d}y}{\mathrm{d}x}+xy=x$, with $y(0)=2$

d) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{y+1}{x+1}$, with $y(0)=1$

e) $\dfrac{\mathrm{d}y}{\mathrm{d}x}+y\cot x=2\csc x$, with $y(\pi/2)=1$

4. A particle of mass m is stationary at time $t=0$ and subject to a force $F(t)=F_0\sin^2\omega t$.
   a) Set up a differential equation to describe the motion.
   b) Let $v=\mathrm{d}x/\mathrm{d}t$, where $x(t)$ denotes the displacement of the particle. By rewriting your equation in part (a) in terms of v if necessary, find an expression for $v(t)$.
   c) Hence, or otherwise, find an expression for $x(t)$.

5. A particle of mass m and with initial velocity v_0 is slowed by a frictional force $F=-b\mathrm{e}^{\alpha v}$, where v denotes its velocity.
   a) Set up a differential equation to describe the motion.
   b) Find the time and distance required for the particle to come to a stop.