FIRST YEAR CALCULUS

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Chapter 16

SECOND ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

16.1. Introduction

The general linear second order ordinary differential equation is the equation

$$a_0(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x}+a_2(x)y=f(x), \tag{1}$$

where $a_0(x), a_1(x), a_2(x)$ and $f(x)$ are given functions. Here we are primarily concerned with (1) only when the coefficients $a_0(x), a_1(x), a_2(x)$ are constants and hence independent of x. We therefore study ordinary differential equations of the type

$$a_0\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+a_1\frac{\mathrm{d}y}{\mathrm{d}x}+a_2y=f(x), \tag{2}$$

where a_0, a_1, a_2 are constants, and where $f(x)$ is a given function.

If the function $f(x)$ on the right hand side of (2) is identically zero, then we say that the ordinary differential equation (2) is homogeneous. If the function $f(x)$ on the right hand side of (2) is not identically zero, then we say that the ordinary differential equation

$$a_0\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+a_1\frac{\mathrm{d}y}{\mathrm{d}x}+a_2y=0 \tag{3}$$

is the reduced equation of (2).

16.2. The Homogeneous Case

In this section, we consider the homogeneous ordinary differential equation (3).

Suppose that y_1 and y_2 are two independent solutions of the equation (3), so that

$$a_0\frac{\mathrm{d}^2y_1}{\mathrm{d}x^2}+a_1\frac{\mathrm{d}y_1}{\mathrm{d}x}+a_2y_1=0 \qquad\text{and}\qquad a_0\frac{\mathrm{d}^2y_2}{\mathrm{d}x^2}+a_1\frac{\mathrm{d}y_2}{\mathrm{d}x}+a_2y_2=0.$$

We consider the linear combination

$$y=C_1y_1+C_2y_2, \tag{4}$$

where C_1 and C_2 are arbitrary constants. Then y is clearly also a solution of (3), for

$$\begin{aligned}a_0\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+a_1\frac{\mathrm{d}y}{\mathrm{d}x}+a_2y&=a_0\left(C_1\frac{\mathrm{d}^2y_1}{\mathrm{d}x^2}+C_2\frac{\mathrm{d}^2y_2}{\mathrm{d}x^2}\right)+a_1\left(C_1\frac{\mathrm{d}y_1}{\mathrm{d}x}+C_2\frac{\mathrm{d}y_2}{\mathrm{d}x}\right)+a_2(C_1y_1+C_2y_2)\\&=C_1\left(a_0\frac{\mathrm{d}^2y_1}{\mathrm{d}x^2}+a_1\frac{\mathrm{d}y_1}{\mathrm{d}x}+a_2y_1\right)+C_2\left(a_0\frac{\mathrm{d}^2y_2}{\mathrm{d}x^2}+a_1\frac{\mathrm{d}y_2}{\mathrm{d}x}+a_2y_2\right)=0.\end{aligned}$$

Since (4) contains two arbitrary constants, it is reasonable to take this as the general solution of the equation (3). It remains to find two independent solutions of the equation (3).

Let us try a solution of the form

$$y=\mathrm{e}^{\lambda x}, \tag{5}$$

where $\lambda\in\mathbb{R}$. Then clearly

$$(a_0\lambda^2+a_1\lambda+a_2)\mathrm{e}^{\lambda x}=0.$$

Since $\mathrm{e}^{\lambda x}\neq 0$, we must have

$$a_0\lambda^2+a_1\lambda+a_2=0. \tag{6}$$

This is called the characteristic polynomial or auxiliary equation of the homogeneous equation (3).

It follows that (5) is a solution of the homogeneous equation (3) whenever λ satisfies the auxiliary equation (6). Suppose that λ_1 and λ_2 are the two roots of (6). Then

$$y_1=\mathrm{e}^{\lambda_1x} \qquad\text{and}\qquad y_2=\mathrm{e}^{\lambda_2x}$$

are both solutions of the homogeneous equation (3). It follows that the general solution of the homogeneous equation (3) is given by

$$y=C_1\mathrm{e}^{\lambda_1x}+C_2\mathrm{e}^{\lambda_2x}, \tag{7}$$

where C_1 and C_2 are arbitrary constants.

EXAMPLE 16.2.1. Suppose that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4\frac{\mathrm{d}y}{\mathrm{d}x}+3y=0.$$

Then the auxiliary equation is given by $\lambda^2+4\lambda+3=0$, with roots $\lambda_1=-3$ and $\lambda_2=-1$. It follows that the general solution of the equation is given by

$$y=C_1\mathrm{e}^{-3x}+C_2\mathrm{e}^{-x}.$$

Example 16.2.2. Suppose that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4y=0.$$

Then the auxiliary equation is given by $\lambda^2+4=0$, with roots $\lambda_1=2\mathrm{i}$ and $\lambda_2=-2\mathrm{i}$. It follows that the general solution of the equation is given by

$$\begin{aligned} y&=B_1\mathrm{e}^{2\mathrm{i}x}+B_2\mathrm{e}^{-2\mathrm{i}x}=B_1(\cos 2x+\mathrm{i}\sin 2x)+B_2(\cos 2x-\mathrm{i}\sin 2x)\\ &=(B_1+B_2)\cos 2x+\mathrm{i}(B_1-B_2)\sin 2x=C_1\cos 2x+C_2\sin 2x. \end{aligned}$$

Example 16.2.3. Suppose that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\frac{\mathrm{d}y}{\mathrm{d}x}+10y=0.$$

Then the auxiliary equation is given by $\lambda^2-2\lambda+10=0$, with roots $\lambda_1=1+3\mathrm{i}$ and $\lambda_2=1-3\mathrm{i}$. It follows that the general solution of the equation is given by

$$\begin{aligned} y&=B_1\mathrm{e}^{(1+3\mathrm{i})x}+B_2\mathrm{e}^{(1-3\mathrm{i})x}=\mathrm{e}^x\left(B_1\mathrm{e}^{3\mathrm{i}x}+B_2\mathrm{e}^{-3\mathrm{i}x}\right)\\ &=\mathrm{e}^x\left(B_1(\cos 3x+\mathrm{i}\sin 3x)+B_2(\cos 3x-\mathrm{i}\sin 3x)\right)\\ &=\mathrm{e}^x\left((B_1+B_2)\cos 3x+\mathrm{i}(B_1-B_2)\sin 3x\right)\\ &=C_1\mathrm{e}^x\cos 3x+C_2\mathrm{e}^x\sin 3x. \end{aligned}$$

The method works well provided that $\lambda_1\neq\lambda_2$. However, if $\lambda_1=\lambda_2$, then (7) does not qualify as the general solution of the homogeneous equation (3), as it contains only one arbitrary constant. We therefore try for a solution of the form

$$y=u\mathrm{e}^{\lambda x}, \tag{8}$$

where u is a function of x, and where λ is the repeated root of the auxiliary equation (6). Substituting (8) into (3), we obtain

$$a_0\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+(2a_0\lambda+a_1)\frac{\mathrm{d}u}{\mathrm{d}x}+(a_0\lambda^2+a_1\lambda+a_2)u=0. \tag{9}$$

Note now that $a_0\lambda^2+a_1\lambda+a_2=0$. Also, since λ is a repeated root, we must have $2\lambda=-a_1/a_0$. It follows that the equation (9) is of the form

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2}=0, \tag{10}$$

so that

$$u=C_1+C_2x,$$

where C_1 and C_2 are arbitrary constants. It follows that the general solution of the equation (3) in this case is given by

$$y=(C_1+C_2x)\mathrm{e}^{\lambda x}, \tag{11}$$

where λ is the repeated root of the auxiliary equation (6), and where C_1 and C_2 are arbitrary constants.

EXAMPLE 16.2.4. Suppose that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 0.$$

Then the auxiliary equation is given by $\lambda^2 - 6\lambda + 9 = 0$, with repeated roots $\lambda = 3$. It follows that the general solution of the equation is given by

$$y = (C_1 + C_2 x)\mathrm{e}^{3x},$$

where C_1 and C_2 are arbitrary constants.

16.3. An Analogy

In this section, we make a digression, and consider the following two problems in coordinate geometry.

Consider the line

$$2x - y = 5 \tag{12}$$

on the xy-plane. Suppose that we would like to describe all the points on this line in terms of a parameter t. We may approach this task in the following way. Consider first of all the line

$$2x - y = 0 \tag{13}$$

through the origin. Then it is easy to see that the point $(1, 2)$ lies on the line (13), and that any point on this line is of the form $t(1, 2)$, where $t \in \mathbb{R}$, and vice versa. We have therefore obtained the general solution of all points on the line (13). Now observe that the point $(3, 1)$ lies on the line (12), and that any point on the line (12) can be described by $t(1, 2) + (3, 1)$, where $t \in \mathbb{R}$, and vice versa. We have therefore obtained the general solution of all points on the line (12).

Consider the plane

$$2x - y + 4z = 10 \tag{14}$$

on the xyz-space. Suppose that we would like to describe all the points on this plane in terms of two parameters t and u. We may approach this task in the following way. Consider first of all the plane

$$2x - y + 4z = 0 \tag{15}$$

through the origin. Then it is easy to see that the points $(1, 2, 0)$ and $(2, 0, -1)$ lie on the plane (15), and that any point on this plane is of the form $t(1, 2, 0) + u(2, 0, -1)$, where $t, u \in \mathbb{R}$, and vice versa. We have therefore obtained the general solution of all points on the plane (15). Now observe that the point $(2, 2, 2)$ lies on the plane (14), and that any point on the plane (14) can be described by $t(1, 2, 0) + u(2, 0, -1) + (2, 2, 2)$, where $t, u \in \mathbb{R}$, and vice versa. We have therefore obtained the general solution of all points on the plane (14).

Now we can think of (13) and (15) as the "reduced equations" of the lines (12) and (14) respectively. Note that we have obtained the general solutions of (13) and (15), while solving (12) and (14) only for a particular solution in each case.

In the next section, we shall mimic this argument.

16.4. The Non-Homogeneous Case

In this section, we consider the non-homogeneous ordinary differential equation (2).

Of course, we expect the general solution of the equation (2) to have two arbitrary constants. Suppose that y_p is a particular solution of the ordinary differential equation (2), and suppose that y_c is the general solution of the reduced equation (3). Then

$$y = y_c + y_p \tag{16}$$

is a solution of the ordinary differential equation (2), for

$$\begin{aligned} a_0\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + a_1\frac{\mathrm{d}y}{\mathrm{d}x} + a_2y &= a_0\left(\frac{\mathrm{d}^2y_c}{\mathrm{d}x^2} + \frac{\mathrm{d}^2y_p}{\mathrm{d}x^2}\right) + a_1\left(\frac{\mathrm{d}y_c}{\mathrm{d}x} + \frac{\mathrm{d}y_p}{\mathrm{d}x}\right) + a_2(y_c + y_p) \\ &= \left(a_0\frac{\mathrm{d}^2y_c}{\mathrm{d}x^2} + a_1\frac{\mathrm{d}y_c}{\mathrm{d}x} + a_2y_c\right) + \left(a_0\frac{\mathrm{d}^2y_p}{\mathrm{d}x^2} + a_1\frac{\mathrm{d}y_p}{\mathrm{d}x} + a_2y_p\right) = 0 + f(x). \end{aligned}$$

It is therefore reasonable to say that (16) is the general solution of the ordinary differential equation (2).

The term y_c is known as the complementary function and the term y_p is known as a particular integral. A particular integral can be any solution of the ordinary differential equation (2); since the difference of any two particular integrals must satisfy the reduced equation (3) and is therefore implicitly taken care of by the complementary function.

To solve the non-homogeneous equation (2), it remains to find a particular integral y_p.

Example 16.4.1. Consider the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = 3x. \tag{17}$$

To obtain the complementary function, we first investigate the reduced equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = 0.$$

This has auxiliary equation $\lambda^2+1 = 0$, with roots $\lambda_1 = \mathrm{i}$ and $\lambda_2 = -\mathrm{i}$. It follows that the complementary function is given by

$$\begin{aligned} y_c &= B_1\mathrm{e}^{\mathrm{i}x} + B_2\mathrm{e}^{-\mathrm{i}x} = B_1(\cos x + \mathrm{i}\sin x) + B_2(\cos x - \mathrm{i}\sin x) \\ &= (B_1 + B_2)\cos x + \mathrm{i}(B_1 - B_2)\sin x = C_1\cos x + C_2\sin x. \end{aligned}$$

To solve the equation (17), it remains to find a particular integral y_p of (17). By inspection, we see that we can take $y_p = 3x$. It follows that

$$y = y_c + y_p = C_1\cos x + C_2\sin x + 3x$$

is the general solution of the equation (17).

16.5. The Method of Undetermined Coefficients

However, in many other cases, we may not be so lucky. To find a particular integral for the equation (2), we therefore need some ideas and/or information in order to make educated guesses. Such information is provided by the given function $f(x)$ and by the complementary function.

In this section, we are concerned with the question of finding particular integrals of differential equations of the type (2), where $a_0, a_1, a_2 k$ are constants, and where $f(x)$ is a given function.

The method of undetermined coefficients is based on assuming a trial form for the particular integral y_p of (2) which depends on the form of the function $f(x)$ and which contains a number of arbitrary constants. This trial function is then substituted into the differential equation (2) and the constants are chosen to make this a solution.

The basic trial forms are given in the table below (a denotes a constant in the expression of $f(x)$ and A (with or without subscripts) denotes a constant to be determined):

$f(x)$	trial y_p	$f(x)$	trial y_p
a	A	$a \sin bx$	$A_1 \cos bx + A_2 \sin bx$
ax	$A_0 + A_1 x$	$a \cos bx$	$A_1 \cos bx + A_2 \sin bx$
ax^2	$A_0 + A_1 x + A_2 x^2$	$a\mathrm{e}^{kx} \sin bx$	$\mathrm{e}^{kx}(A_1 \cos bx + A_2 \sin bx)$
$ax^m \ (m \in \mathbb{N})$	$A_0 + A_1 x + \ldots + A_m x^m$	$a\mathrm{e}^{kx} \cos bx$	$\mathrm{e}^{kx}(A_1 \cos bx + A_2 \sin bx)$
$a\mathrm{e}^{bx} \ (r \in \mathbb{R})$	$A\mathrm{e}^{bx}$	$ax^m \mathrm{e}^{bx}$	$\mathrm{e}^{bx}(A_0 + A_1 x + \ldots + A_m x^m)$

Example 16.5.1. Suppose that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = 30\mathrm{e}^{2x}.$$

It has been shown in Example 16.2.1 that the reduced equation has complementary function

$$y_c = C_1 \mathrm{e}^{-3x} + C_2 \mathrm{e}^{-x}.$$

For a particular integral, we try $y_p = A\mathrm{e}^{2x}$. Substituting into the equation, we obtain

$$\frac{\mathrm{d}^2 y_p}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y_p}{\mathrm{d}x} + 3y_p = (4A + 8A + 3A)\mathrm{e}^{2x} = 15A\mathrm{e}^{2x} = 30\mathrm{e}^{2x}$$

if $A = 2$. Hence

$$y = y_c + y_p = C_1 \mathrm{e}^{-3x} + C_2 \mathrm{e}^{-x} + 2\mathrm{e}^{2x}.$$

Example 16.5.2. Suppose that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4y = 6 \cos x.$$

It has been shown in Example 16.2.2 that the reduced equation has complementary function

$$y_c = C_1 \cos 2x + C_2 \sin 2x.$$

For a particular integral, we try $y_p = A_1 \cos x + A_2 \sin x$. Substituting into the equation, we obtain

$$\frac{\mathrm{d}^2 y_p}{\mathrm{d}x^2} + 4y_p = (-A_1 + 4A_1) \cos x + (-A_2 + 4A_2) \sin x = 3A_1 \cos x + 3A_2 \sin x = 6 \cos x$$

if $A_1 = 2$ and $A_2 = 0$. Hence

$$y = y_c + y_p = C_1 \cos 2x + C_2 \sin 2x + 2 \cos x.$$

EXAMPLE 16.5.3. Suppose that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 40\mathrm{e}^{-x}\cos 3x.$$

It has been shown in Example 16.2.3 that the reduced equation has complementary function

$$y_c = C_1\mathrm{e}^x \cos 3x + C_2\mathrm{e}^x \sin 3x.$$

For a particular integral, we try $y_p = \mathrm{e}^{-x}(A_1 \cos 3x + A_2 \sin 3x)$. Substituting into the equation, we obtain

$$\frac{\mathrm{d}^2y_p}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y_p}{\mathrm{d}x} + 10y_p = \mathrm{e}^{-x}((4A_1 - 12A_2)\cos 3x + (12A_1 + 4A_2)\sin 3x) = 40\mathrm{e}^{-x}\cos 3x$$

if $4A_1 - 12A_2 = 40$ and $12A_1 + 4A_2 = 0$; in other words, if $A_1 = 1$ and $A_2 = -3$. Hence

$$y = y_c + y_p = C_1\mathrm{e}^x \cos 3x + C_2\mathrm{e}^x \sin 3x + \mathrm{e}^{-x}\cos 3x - 3\mathrm{e}^{-x}\sin 3x.$$

16.6. Lifting the Trial Functions

What we have discussed so far in Section 16.5 may not work in situations where the standard trial functions are too intimately related to the complementary functions. In such cases, we need to modify the trial functions. Through the use of a few examples, we shall try to understand why the standard trial functions do not work in these situations and discuss how we may modify them to enable us to find particular integrals in a similar way as before.

EXAMPLE 16.6.1. Suppose that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = 4\mathrm{e}^{-x}.$$

It has been shown in Example 16.2.1 that the reduced equation has complementary function

$$y_c = C_1\mathrm{e}^{-3x} + C_2\mathrm{e}^{-x},$$

where C_1 and C_2 are arbitrary constants. For a particular integral, we try $y_p = A\mathrm{e}^{-x}$. Substituting into the equation, we obtain

$$\frac{\mathrm{d}^2y_p}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y_p}{\mathrm{d}x} + 3y_p = (A - 4A + 3A)\mathrm{e}^{-x} = 0 \neq 4\mathrm{e}^{-x}$$

for any A. In fact, this is no coincidence. Note that if we take $C_1 = 0$ and $C_2 = A$, then the complementary function y_c becomes our trial function! No wonder the method does not work. Now try instead

$$y_p = Ax\mathrm{e}^{-x}. \tag{18}$$

Substituting into the equation, we obtain

$$\frac{\mathrm{d}^2y_p}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y_p}{\mathrm{d}x} + 3y_p = 2A\mathrm{e}^{-x} = 4\mathrm{e}^{-x}$$

if $A = 2$. Hence

$$y = y_c + y_p = C_1\mathrm{e}^{-3x} + C_2\mathrm{e}^{-x} + 2x\mathrm{e}^{-x}.$$

EXAMPLE 16.6.2. Suppose that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4y=4\sin 2x.$$

It has been shown in Example 16.2.2 that the reduced equation has complementary function

$$y_c=C_1\cos 2x+C_2\sin 2x,$$

where C_1 and C_2 are arbitrary constants. For a particular integral, we try $y_p=A_1\cos 2x+A_2\sin 2x$. Substituting into the equation, we obtain

$$\frac{\mathrm{d}^2y_p}{\mathrm{d}x^2}+4y_p=(-4A_1+4A_1)\cos 2x+(-4A_2+4A_2)\sin x=0\neq 4\sin 2x$$

for any A. In fact, this is no coincidence. Note that if we take $C_1=A_1$ and $C_2=A_2$, then the complementary function y_c becomes our trial function! Now try instead

$$y_p=x(A_1\cos 2x+A_2\sin 2x). \tag{19}$$

Substituting into the equation, we obtain

$$\frac{\mathrm{d}^2y_p}{\mathrm{d}x^2}+4y_p=4A_2\cos 2x-4A_1\sin 2x=4\sin 2x$$

if $A_1=-1$ and $A_2=0$. Hence

$$y=y_c+y_p=C_1\cos 2x+C_2\sin 2x-x\cos 2x.$$

EXAMPLE 16.6.3. Suppose that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\frac{\mathrm{d}y}{\mathrm{d}x}+10y=6\mathrm{e}^x\sin 3x.$$

It has been shown in Example 16.2.3 that the reduced equation has complementary function

$$y_c=C_1\mathrm{e}^x\cos 3x+C_2\mathrm{e}^x\sin 3x,$$

where C_1 and C_2 are arbitrary constants. For a particular integral, we try $y_p=\mathrm{e}^x(A_1\cos 3x+A_2\sin 3x)$. This is bound to fail, for if we take $C_1=A_1$ and $C_2=A_2$, then the complementary function y_c becomes our trial function! Now try instead

$$y_p=x\mathrm{e}^x(A_1\cos 3x+A_2\sin 3x). \tag{20}$$

Substituting into the equation, we obtain

$$\frac{\mathrm{d}^2y_p}{\mathrm{d}x^2}-2\frac{\mathrm{d}y_p}{\mathrm{d}x}+10y_p=6A_2\mathrm{e}^x\cos 3x-6A_1\mathrm{e}^x\sin 3x=6\mathrm{e}^x\sin 3x$$

if $A_1=-1$ and $A_2=0$. Hence

$$y=y_c+y_p=C_1\mathrm{e}^x\cos 3x+C_2\mathrm{e}^x\sin 3x-x\mathrm{e}^x\cos 3x.$$

The next example involves a complementary function which itself has already been “lifted”.

Example 16.6.4. Suppose that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 6\mathrm{e}^{3x}.$$

It has been shown in Example 16.2.4 that the reduced equation has complementary function

$$y_c = (C_1 + C_2x)\mathrm{e}^{3x},$$

where C_1 and C_2 are arbitrary constants. For a particular integral, we try $y_p = A\mathrm{e}^{3x}$. This is bound to fail, for if we take $C_1 = A$ and $C_2 = 0$, then the complementary function y_c becomes our trial function! Now try instead $y_p = Ax\mathrm{e}^{3x}$. This again is bound to fail, for if we take $C_1 = 0$ and $C_2 = A$, then the complementary function y_c becomes our trial function! We therefore try

$$y_p = Ax^2\mathrm{e}^{3x}. \tag{21}$$

Substituting into the equation, we obtain

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 2A\mathrm{e}^{3x} = 6\mathrm{e}^{3x}$$

if $A = 3$. Hence

$$y = y_c + y_p = (C_1 + C_2x)\mathrm{e}^{3x} + 3x^2\mathrm{e}^{3x}.$$

In general, all we need to do when the usual trial function forms part of the complementary function is to "lift our usual trial function over the complementary function" by multiplying the usual trial function by a power of x. This power should be as small as possible, as overlifting can cause difficulties, as shown by the example below.

Example 16.6.5. Let us return to Example 16.5.1, where we considered the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = 30\mathrm{e}^{2x}.$$

There we successfully used the trial function $y_p = A\mathrm{e}^{2x}$. Suppose now that we lift the trial function unnecessarily, and try instead $y_p = Ax\mathrm{e}^{2x}$. Substituting into the equation, we obtain

$$\frac{\mathrm{d}^2y_p}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y_p}{\mathrm{d}x} + 3y_p = 15Ax\mathrm{e}^{2x} + 8A\mathrm{e}^{2x} = 30\mathrm{e}^{2x}$$

if $15A = 0$ and $8A = 30$, clearly impossible.

16.7. Further Examples

In this section, we briefly describe some finer points in the application of the method of undetermined coefficients. We illustrate by two examples. The reader is expected to complete the details – some hard work is required here!

EXAMPLE 16.7.1. Suppose that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = 6\cos x - 4\sin 2x.$$

It has been shown in Example 16.2.2 that the reduced equation has complementary function

$$y_c = C_1 \cos 2x + C_2 \sin 2x,$$

where C_1 and C_2 are arbitrary constants. For a particular integral, we try

$$y_p = (A_1 \cos x + A_2 \sin x) + (A_3 \cos 2x + A_4 \sin 2x).$$

Substituting into the equation (the reader must try this) and equating coefficients, we find that we can equate coefficients for $\cos x$ and $\sin x$, but not for $\cos 2x$ and $\sin 2x$. This is no coincidence, for $A_3 \cos 2x + A_4 \sin 2x$ resembles the complementary function, so that we must lift this part. The correct trial function is therefore

$$y_p = (A_1 \cos x + A_2 \sin x) + x(A_3 \cos 2x + A_4 \sin 2x). \qquad (22)$$

Substituting into the equation (the reader again must try this) and equating coefficients, we find that $A_1 = 2$, $A_2 = 0$, $A_3 = 1$ and $A_4 = 0$. Hence

$$y = y_c + y_p = C_1 \cos 2x + C_2 \sin 2x + 2\cos x + x\cos 2x.$$

EXAMPLE 16.7.2. Suppose that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 16\mathrm{e}^x \sin x + 40\mathrm{e}^{-x}\cos 3x - 6\mathrm{e}^x \sin 3x.$$

It has been shown in Example 16.2.3 that the reduced equation has complementary function

$$y_c = C_1\mathrm{e}^x \cos 3x + C_2\mathrm{e}^x \sin 3x,$$

where C_1 and C_2 are arbitrary constants. For a particular integral, we try

$$y_p = \mathrm{e}^x(A_1 \cos x + A_2 \sin x) + \mathrm{e}^{-x}(A_3 \cos 3x + A_4 \sin 3x) + \mathrm{e}^x(A_5 \cos 3x + A_6 \sin 3x).$$

Substituting into the equation (the reader must try this) and equating coefficients, we find that we can equate coefficients for $\mathrm{e}^x \cos x$, $\mathrm{e}^x \sin x$, $\mathrm{e}^{-x} \cos 3x$ and $\mathrm{e}^{-x} \sin 3x$, but not for $\mathrm{e}^x \cos 3x$ and $\mathrm{e}^x \sin 3x$. This is no coincidence, for $\mathrm{e}^x(A_5 \cos 3x + A_6 \sin 3x)$ resembles the complementary function, so that we must lift this part. The correct trial function is therefore

$$y_p = \mathrm{e}^x(A_1 \cos x + A_2 \sin x) + \mathrm{e}^{-x}(A_3 \cos 3x + A_4 \sin 3x) + x\mathrm{e}^x(A_5 \cos 3x + A_6 \sin 3x). \qquad (23)$$

Substituting into the equation (the reader again must try this) and equating coefficients, we find that $A_1 = 0$, $A_2 = 2$, $A_3 = 1$, $A_4 = -3$, $A_5 = 1$ and $A_6 = 0$. Hence

$$y = y_c + y_p = C_1\mathrm{e}^x \cos 3x + C_2\mathrm{e}^x \sin 3x + 2\mathrm{e}^x \sin x + \mathrm{e}^{-x}(\cos 3x - 3\sin 3x) + x\mathrm{e}^x \cos 3x.$$

16.8. A More Systematic Approach for Particular Integrals

In this section, we describe a technique which takes all the guessing out of the method of undetermined coefficients, and gives us a better understanding of the lifting technique. To understand this technique, we need to extend our discussion of homogeneous equations to higher order. This presents very little extra difficulty.

Consider again the homogeneous linear differential equation

$$a_0\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+a_1\frac{\mathrm{d}y}{\mathrm{d}x}+a_2y=0.$$

We can rewrite this equation in the form

$$\mathcal{L}(D)y=a_0D^2y+a_1Dy+a^2y=0,$$

where $\mathcal{L}(D)=a_0D^2+a_1D+a_2$ is a quadratic polynomial of the differential operator D, where

$$D^ky=\frac{\mathrm{d}^ky}{\mathrm{d}x^k}\quad\text{for every } k\in\mathbb{N}.$$

A non-homogeneous linear differential equation

$$a_0\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+a_1\frac{\mathrm{d}y}{\mathrm{d}x}+a_2y=f(x)$$

can now be written in the form

$$\mathcal{L}(D)y=f(x).$$

Suppose that the function $f(x)$ satisfies a homogeneous linear differential equation $\mathcal{M}(D)f=0$, where $\mathcal{M}(D)$ is a polynomial of the differential operator D, preferably of smallest degree. Then we must have

$$\mathcal{M}(D)\mathcal{L}(D)y=0.$$

But this is a homogeneous linear differential equation of higher order, and can be solved by using the roots of the auxiliary equation

$$\mathcal{M}(\lambda)\mathcal{L}(\lambda)=0$$

in a way similar to the discussion in Section 16.2 for homogeneous second order linear differential equations.

We illustrate our technique by revisiting a few examples.

Example 16.8.1. Consider the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4\frac{\mathrm{d}y}{\mathrm{d}x}+3y=30\mathrm{e}^{2x},$$

first considered in Example 16.5.1. Recall that the reduced equation has complementary function

$$y_c=C_1\mathrm{e}^{-3x}+C_2\mathrm{e}^{-x},$$

where C_1 and C_2 are arbitrary constants, and that for the particular integral, we try $y_p=A\mathrm{e}^{2x}$. Note that the original equation can be written in the form $\mathcal{L}(D)y=f(x)$, where $\mathcal{L}(D)=D^2+4D+3$ and $f(x)=30\mathrm{e}^{2x}$. It is not difficult to see that $\mathcal{M}(D)f=0$, where $\mathcal{M}(D)=D-2$, as the solution e^{2x} corresponds to $\lambda=2$. It follows that

$$\mathcal{M}(D)\mathcal{L}(D)y=(D^2+4D+3)(D-2)y=0,$$

with auxiliary equation $(\lambda^2+4\lambda+3)(\lambda-2)=(\lambda+3)(\lambda+1)(\lambda-2)=0$, with roots $\lambda=-3$, $\lambda=-1$ and $\lambda=2$. Hence

$$y=C_1\mathrm{e}^{-3x}+C_2\mathrm{e}^{-x}+C_3\mathrm{e}^{2x}.$$

But then we know that $C_1\mathrm{e}^{-3x}+C_2\mathrm{e}^{-x}$ is the complementary function of the original equation. Hence $C_3\mathrm{e}^{2x}$ must be the trial function for the particular integral. We now proceed as in Example 16.5.1 to conclude that we must take $C_3=2$, so that

$$y=C_1\mathrm{e}^{-3x}+C_2\mathrm{e}^{-x}+2\mathrm{e}^{2x}.$$

Example 16.8.2. Consider the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4\frac{\mathrm{d}y}{\mathrm{d}x}+3y=4\mathrm{e}^{-x},$$

first considered in Example 16.6.1. Recall that the reduced equation has complementary function

$$y_c=C_1\mathrm{e}^{-3x}+C_2\mathrm{e}^{-x},$$

where C_1 and C_2 are arbitrary constants. Note that the original equation can be written in the form $\mathcal{L}(D)y=f(x)$, where $\mathcal{L}(D)=D^2+4D+3$ and $f(x)=4\mathrm{e}^{-x}$. It is not difficult to see that $\mathcal{M}(D)f=0$, where $\mathcal{M}(D)=D+1$, as the solution e^{-x} corresponds to $\lambda=-1$. It follows that

$$\mathcal{M}(D)\mathcal{L}(D)y=(D^2+4D+3)(D+1)y=0,$$

with auxiliary equation $(\lambda^2+4\lambda+3)(\lambda+1)=(\lambda+3)(\lambda+1)^2=0$, with roots $\lambda=-3$ and $\lambda=-1$ (twice). Hence

$$y=C_1\mathrm{e}^{-3x}+(C_2+C_3x)\mathrm{e}^{-x}.$$

But then we know that $C_1\mathrm{e}^{-3x}+C_2\mathrm{e}^{-x}$ is the complementary function of the original equation. Hence $C_3x\mathrm{e}^{-x}$ must be the trial function for the particular integral; see (18). We now proceed as in Example 16.6.1 to conclude that we must take $C_3=2$, so that

$$y=C_1\mathrm{e}^{-3x}+C_2\mathrm{e}^{-x}+2x\mathrm{e}^{-x}.$$

Example 16.8.3. Consider the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4y=4\sin 2x,$$

first considered in Example 16.6.2. Recall that the reduced equation has complementary function

$$y_c=C_1\cos 2x+C_2\sin 2x,$$

where C_1 and C_2 are arbitrary constants. Note that the original equation can be written in the form $\mathcal{L}(D)y=f(x)$, where $\mathcal{L}(D)=D^2+4$ and $f(x)=4\sin 2x$. It is not difficult to see that $\mathcal{M}(D)f=0$, where $\mathcal{M}(D)=D^2+4$, as the solution $\sin 2x$ corresponds to $\lambda=\pm 2\mathrm{i}$. It follows that

$$\mathcal{M}(D)\mathcal{L}(D)y=(D^2+4)(D^2+4)y=0,$$

with auxiliary equation $(\lambda^2+4)^2=0$, with roots $\lambda=2\mathrm{i}$ (twice) and $\lambda=-2\mathrm{i}$ (twice). Hence

$$y=(C_1+C_2x)\cos 2x+(C_3+C_4x)\sin 2x.$$

But then we know that $C_1 \cos 2x + C_3 \sin 2x$ is the complementary function of the original equation. Hence $C_2 x \cos 2x + C_4 x \sin 2x$ must be the trial function for the particular integral; see (19). We now proceed as in Example 16.6.2 to conclude that we must take $C_2 = -1$ and $C_4 = 0$, so that

$$y = C_1 \cos 2x + C_3 \sin 2x - x \cos 2x.$$

EXAMPLE 16.8.4. Consider the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 6\mathrm{e}^x \sin 3x,$$

first considered in Example 16.6.3. Recall that the reduced equation has complementary function

$$y_c = C_1 \mathrm{e}^x \cos 3x + C_2 \mathrm{e}^x \sin 3x,$$

where C_1 and C_2 are arbitrary constants. Note that the original equation can be written in the form $\mathcal{L}(D)y = f(x)$, where $\mathcal{L}(D) = D^2 - 2D + 10$ and $f(x) = 6\mathrm{e}^x \sin 3x$. It is not difficult to see that $\mathcal{M}(D)f = 0$, where $\mathcal{M}(D) = D^2 - 2D + 10$, as the solution $\mathrm{e}^x \sin 3x$ corresponds to $\lambda = 1 \pm 3\mathrm{i}$. It follows that

$$\mathcal{M}(D)\mathcal{L}(D)y = (D^2 - 2D + 10)(D^2 - 2D + 10)y = 0,$$

with auxiliary equation $(\lambda^2 - 2\lambda + 10)^2 = 0$, with roots $\lambda = 1 + 3\mathrm{i}$ (twice) and $\lambda = 1 - 3\mathrm{i}$ (twice). Hence

$$y = (C_1 + C_2 x)\mathrm{e}^x \cos 3x + (C_3 + C_4 x)\mathrm{e}^x \sin 3x.$$

But then we know that $C_1 \mathrm{e}^x \cos 3x + C_3 \mathrm{e}^x \sin 3x$ is the complementary function of the original equation. Hence $C_2 x\mathrm{e}^x \cos 3x + C_4 x\mathrm{e}^x \sin 3x$ must be the trial function for the particular integral; see (20). We now proceed as in Example 16.6.3 to conclude that we must take $C_2 = -1$ and $C_4 = 0$, so that

$$y = C_1 \mathrm{e}^x \cos 3x + C_3 \mathrm{e}^x \sin 3x - x\mathrm{e}^x \cos 3x.$$

EXAMPLE 16.8.5. Consider the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 6\mathrm{e}^{3x},$$

first considered in Example 16.6.4. Recall that the reduced equation has complementary function

$$y_c = (C_1 + C_2 x)\mathrm{e}^{3x},$$

where C_1 and C_2 are arbitrary constants. Note that the original equation can be written in the form $\mathcal{L}(D)y = f(x)$, where $\mathcal{L}(D) = D^2 - 6D + 9$ and $f(x) = 6\mathrm{e}^{3x}$. It is not difficult to see that $\mathcal{M}(D)f = 0$, where $\mathcal{M}(D) = D - 3$, as the solution e^{3x} corresponds to $\lambda = 3$. It follows that

$$\mathcal{M}(D)\mathcal{L}(D)y = (D^2 - 6D + 9)(D - 3)y = 0,$$

with auxiliary equation $(\lambda^2 - 6\lambda + 9)(\lambda - 3) = (\lambda - 3)^3$, with root $\lambda = 3$ (three times). One can show, analogous to (8)–(11), that

$$y = (C_1 + C_2 x + C_3 x^2)\mathrm{e}^{3x}.$$

But then we know that $(C_1 + C_2 x)\mathrm{e}^{3x}$ is the complementary function of the original equation. Hence $C_3 x^2 \mathrm{e}^{3x}$ must be the trial function for the particular integral; see (21). We now proceed as in Example 16.6.4 to conclude that we must take $C_3 = 3$, so that

$$y = (C_1 + C_2 x)\mathrm{e}^{3x} + 3x^2 \mathrm{e}^{3x}.$$

Our last two examples in this section involve functions $f(x)$ that are rather complicated, and give a good illustration of the power and versatility of our technique.

EXAMPLE 16.8.6. Consider the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4y=6\cos x-4\sin 2x,$$

first considered in Example 16.7.1. Recall that the reduced equation has complementary function

$$y_c=C_1\cos 2x+C_2\sin 2x,$$

where C_1 and C_2 are arbitrary constants. Note that the original equation can be written in the form $\mathcal{L}(D)y=f(x)$, where $\mathcal{L}(D)=D^2+4$ and $f(x)=6\cos x-4\sin 2x$. We now observe that the solution $\cos x$ corresponds to $\lambda=\pm\mathrm{i}$, while the solution $\sin 2x$ corresponds to $\lambda=\pm 2\mathrm{i}$. Hence $\mathcal{M}(D)f=0$, where $\mathcal{M}(D)=(D^2+1)(D^2+4)$. It follows that

$$\mathcal{M}(D)\mathcal{L}(D)y=(D^2+1)(D^2+4)(D^2+4)y=0,$$

with auxiliary equation $(\lambda^2+1)(\lambda^2+4)^2=0$, with roots $\lambda=\mathrm{i}$, $\lambda=-\mathrm{i}$, $\lambda=2\mathrm{i}$ (twice) and $\lambda=-2\mathrm{i}$ (twice). Hence

$$y=C_1\cos x+C_2\sin x+(C_3+C_4x)\cos 2x+(C_5+C_6x)\sin 2x.$$

But then we know that $C_3\cos 2x+C_5\sin 2x$ is the complementary function of the original equation. Hence $C_1\cos x+C_2\sin x+C_4x\cos 2x+C_6x\sin 2x$ must be the trial function for the particular integral; see (22). One can show that we must have $C_1=2$, $C_2=0$, $C_4=1$ and $C_6=0$, so that

$$y=C_3\cos 2x+C_5\sin 2x+2\cos x+x\cos 2x.$$

EXAMPLE 16.8.7. Consider the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\frac{\mathrm{d}y}{\mathrm{d}x}+10y=16\mathrm{e}^x\sin x+40\mathrm{e}^{-x}\cos 3x-6\mathrm{e}^x\sin 3x.$$

first considered in Example 16.7.2. Recall that the reduced equation has complementary function

$$y_c=C_1\mathrm{e}^x\cos 3x+C_2\mathrm{e}^x\sin 3x,$$

where C_1 and C_2 are arbitrary constants. Note that the original equation can be written in the form $\mathcal{L}(D)y=f(x)$, where $\mathcal{L}(D)=D^2-2D+10$ and $f(x)=16\mathrm{e}^x\sin x+40\mathrm{e}^{-x}\cos 3x-6\mathrm{e}^x\sin 3x$. We now observe that the solution $\mathrm{e}^x\sin x$ corresponds to $\lambda=1\pm\mathrm{i}$, the solution $\mathrm{e}^{-x}\cos 3x$ corresponds to $\lambda=-1\pm 3\mathrm{i}$, while the solution $\mathrm{e}^x\sin 3x$ corresponds to $\lambda=1\pm 3\mathrm{i}$. Hence $\mathcal{M}(D)f=0$, where

$$\mathcal{M}(D)=(D-1+\mathrm{i})(D-1-\mathrm{i})(D+1+3\mathrm{i})(D+1-3\mathrm{i})(D-1+3\mathrm{i})(D-1-3\mathrm{i}).$$

It follows that

$$\mathcal{M}(D)\mathcal{L}(D)y=(D-1+\mathrm{i})(D-1-\mathrm{i})(D+1+3\mathrm{i})(D+1-3\mathrm{i})(D-1+3\mathrm{i})(D-1-3\mathrm{i})(D^2-2D+10)y=0,$$

with auxiliary equation

$$\begin{aligned}&(\lambda-1+\mathrm{i})(\lambda-1-\mathrm{i})(\lambda+1+3\mathrm{i})(\lambda+1-3\mathrm{i})(\lambda-1+3\mathrm{i})(\lambda-1-3\mathrm{i})(\lambda^2-2\lambda+10)\\&\quad=(\lambda-1+\mathrm{i})(\lambda-1-\mathrm{i})(\lambda+1+3\mathrm{i})(\lambda+1-3\mathrm{i})(\lambda-1+3\mathrm{i})^2(\lambda-1-3\mathrm{i})^2=0,\end{aligned}$$

with roots $\lambda = 1 + \mathrm{i}$, $\lambda = 1 - \mathrm{i}$, $\lambda = -1 + 3\mathrm{i}$, $\lambda = -1 - 3\mathrm{i}$, $\lambda = 1 + 3\mathrm{i}$ (twice) and $\lambda = 1 - 3\mathrm{i}$ (twice). Hence

$$y = C_1\mathrm{e}^x \cos x + C_2\mathrm{e}^x \sin x + C_3\mathrm{e}^{-x} \cos 3x + C_4\mathrm{e}^{-x} \sin 3x + (C_5 + C_6x)\mathrm{e}^x \cos 3x + (C_7 + C_8x)\mathrm{e}^x \sin 3x.$$

But then we know that $C_5\mathrm{e}^x \cos 3x + C_7\mathrm{e}^x \sin 3x$ is the complementary function of the original equation. Hence

$$C_1\mathrm{e}^x \cos x + C_2\mathrm{e}^x \sin x + C_3\mathrm{e}^{-x} \cos 3x + C_4\mathrm{e}^{-x} \sin 3x + C_6x\mathrm{e}^x \cos 3x + C_8x\mathrm{e}^x \sin 3x$$

must be the trial function for the particular integral; see (23). One can show that we must have $C_1 = 0$, $C_2 = 2$, $C_3 = 1$, $C_4 = -3$, $C_6 = 1$ and $C_8 = 0$, so that

$$y = C_5\mathrm{e}^x \cos 3x + C_7\mathrm{e}^x \sin 3x + 2\mathrm{e}^x \sin x + \mathrm{e}^{-x} \cos 3x - 3\mathrm{e}^{-x} \sin 3x + x\mathrm{e}^x \cos 3x.$$

16.9. Initial Conditions

In many of the examples of second order linear differential equations we have investigated, the solution is of the form $y = y_c + y_p$, where the complementary function y_c contains two arbitrary constants. If we investigate such equations with given initial conditions, then these two constants no longer remain arbitrary. The initial conditions are usually given in terms of specific values for y and $\mathrm{d}y/\mathrm{d}x$ at $x = 0$.

EXAMPLE 16.9.1. Suppose that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = 4\mathrm{e}^{-x}.$$

It has been shown in Example 16.6.1 that the equation has solution

$$y = C_1\mathrm{e}^{-3x} + C_2\mathrm{e}^{-x} + 2x\mathrm{e}^{-x}. \tag{24}$$

Suppose further that $y = 5$ and $\mathrm{d}y/\mathrm{d}x = -9$ at $x = 0$. Differentiating the equation (24), we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -3C_1\mathrm{e}^{-3x} - C_2\mathrm{e}^{-x} + 2\mathrm{e}^{-x} - 2x\mathrm{e}^{-x}. \tag{25}$$

Substituting the initial conditions into (24) and (25), we obtain respectively

$$5 = C_1 + C_2 \qquad \text{and} \qquad -9 = -3C_1 - C_2 + 2.$$

Hence $C_1 = 3$ and $C_2 = 2$. It follows that

$$y = 3\mathrm{e}^{-3x} + 2\mathrm{e}^{-x} + 2x\mathrm{e}^{-x}.$$

EXAMPLE 16.9.2. Suppose that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 6\mathrm{e}^{3x}.$$

It has been shown in Example 16.6.4 that the equation has solution

$$y = (C_1 + C_2x)\mathrm{e}^{3x} + 3x^2\mathrm{e}^{3x}. \tag{26}$$

Suppose further that $y = 0$ and $\mathrm{d}y/\mathrm{d}x = 1$ at $x = 0$. Differentiating the equation (26), we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ((3C_1 + C_2) + (3C_2 + 6)x + 9x^2)\mathrm{e}^{3x}. \tag{27}$$

Substituting the initial conditions into (26) and (27), we obtain respectively

$$0 = C_1 \qquad \text{and} \qquad 1 = 3C_1 + C_2.$$

Hence $C_1 = 0$ and $C_2 = 1$. It follows that

$$y = x\mathrm{e}^{3x} + 3x^2\mathrm{e}^{3x}.$$

16.10. Summary

Consider the non-homogeneous linear differential equation (2), with given initial conditions. To solve this equation completely, we take the following steps in order:

- Consider the reduced equation (3), and find its general solution y_c by finding the roots of its auxiliary equation (6) and using the formula (7). This solution y_c is called the complementary function. The expression for y_c contains two arbitrary constants C_1 and C_2.
- Find a particular solution y_p of the equation (2) by using, for example, the method of undetermined coefficients, bearing in mind that in this method, the usual trial function may have to be lifted above the complementary function.
- Obtain the general solution of the original equation (2) by calculating $y = y_c + y_p$.
- If initial conditions are given, substitute them into the expression for y obtained from the previous step and into the expression for $\mathrm{d}y/\mathrm{d}x$ obtained by differentiating the expression for y. Then determine the constants C_1 and C_2.

16.11. Application to Problems in Physics

In this section, we study some of the differential equations first discussed in Section 14.3. For the examples in mechanics, it is convenient to use t to denote the independent variable representing time, and to use x as the dependent variable representing displacement.

Example 16.11.1. In Example 14.3.3, we consider a body of mass m fastened to a spring whose constant is k. If we stretch the spring by a distance x, then it exerts a restoring force $F = -kx$. If we neglect friction and assume that there are no other forces, then the equation of motion is given by

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + kx = 0.$$

This is a linear second order homogensous ordinary differential equation with constant coefficients, with auxiliary equation $m\lambda^2 + k = 0$, so that $\lambda = \pm\mathrm{i}\omega_0$, where $\omega_0 = \sqrt{k/m}$. The solution is therefore

$$x = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t,$$

where C_1 and C_2 are absolute constants. We can write $C_1 = C\cos\theta$ and $C_2 = -C\sin\theta$ for some fixed real number θ and positive constant C. Then the solution can be rewritten in the form

$$x = C\cos(\omega_0 t + \theta). \tag{28}$$

This is simple harmonic motion with natural frequency $w_0/2\pi$ and period $2\pi/\omega_0$.

EXAMPLE 16.11.2. In Examples 14.3.3 and 14.3.4, we consider a body of mass m fastened to a spring whose constant is k. If we stretch the spring by a distance x, then it exerts a restoring force $F = -kx$. The motion is also subject to a frictional force proportional to the speed of the body. The equation of motion is given by

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0,$$

where $b > 0$ is a fixed proportionality constant. This is a linear second order homogeneous ordinary differential equation with constant coefficients, with auxiliary equation $m\lambda^2 + b\lambda + k = 0$, so that

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} = -\frac{b}{2m} \pm \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}.$$

We distinguish three cases, discussed separately in the next three examples.

EXAMPLE 16.11.3. Suppose that in Example 16.11.2, we have $k/m > b^2/4m^2$. Then

$$\lambda = -\frac{b}{2m} \pm \mathrm{i}\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = -\gamma \pm \mathrm{i}\omega_1,$$

where $\gamma = b/2m$, $\omega_0 = \sqrt{k/m}$ and $\omega_1 = \sqrt{\omega_0^2 - \gamma^2}$. The solution is therefore

$$x = \mathrm{e}^{-\gamma t}(C_1 \cos\omega_1 t + C_2 \sin\omega_1 t),$$

where C_1 and C_2 are absolute constants. Again we can write $C_1 = C\cos\theta$ and $C_2 = -C\sin\theta$ for some fixed real number θ and positive constant C. Then the solution can be rewritten in the form

$$x = C\mathrm{e}^{-\gamma t}\cos(\omega_1 t + \theta). \qquad (29)$$

This is damped simple harmonic motion with frequency $w_1/2\pi$ and period $2\pi/\omega_1$. The constant γ is called the damping coefficient. Comparing (28) and (29), we observe that the damping coefficient reduced the amplitude from C to $C\mathrm{e}^{-\gamma t}$, while the oscillator now has a frequency $\omega_1/2\pi$ which is less than the natural frequency $w_0/2\pi$ of the undamped oscillator. Here we say that the oscillator is underdamped.

EXAMPLE 16.11.4. Suppose that in Example 16.11.2, we have $k/m < b^2/4m^2$. Then

$$\lambda = -\frac{b}{2m} \pm \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}} = -\gamma \pm \gamma_*,$$

where $\gamma = b/2m$, $\omega_0 = \sqrt{k/m}$ and $\gamma_* = \sqrt{\gamma^2 - \omega_0^2} < \gamma$. The solution is therefore

$$x = C_1\mathrm{e}^{-(\gamma+\gamma_*)t} + C_2\mathrm{e}^{-(\gamma-\gamma_*)t}, \qquad (30)$$

where C_1 and C_2 are absolute constants. The two terms decay exponentially with time, with the first one at a faster rate than the second. Here we say that the oscillator is overdamped.

EXAMPLE 16.11.5. Suppose that in Example 16.11.2, we have $k/m = b^2/4m^2$. Then $\lambda = -\gamma$, where $\gamma = b/2m$. The solution is therefore

$$x = (C_1 + C_2 t)\mathrm{e}^{-\gamma t}, \qquad (31)$$

where C_1 and C_2 are absolute constants. Here we say that the oscillator is critically damped.

EXAMPLE 16.11.6. Suppose that in Example 16.11.2, the body is subject to an additional impressed force $F(t)$. Then, as shown in Example 14.3.5, the equation of motion is given by

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F(t).$$

This is a now linear second order non-homogeneous ordinary differential equation with constant coefficients, with auxiliary equation $m\lambda^2 + b\lambda + k = 0$. The complementary function has already been studied in Examples 16.11.3–16.11.5, and is given by (29), (30) or (31). Here we need to obtain a particular integral. The most important case is that of a sinusoidally oscillating applied force with amplitude F_0 and frequency $\omega/2\pi$, so that

$$F(t) = F_0\cos(\omega t + \theta_0),$$

where θ_0 is a constant specifying the phase of the applied force. Then

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F_0\cos(\omega t + \theta_0). \tag{32}$$

We shall attempt to find a particular solution by complex variable technique. Write $G_0 = F_0\mathrm{e}^{\mathrm{i}\theta_0}$. Then

$$G_0\mathrm{e}^{\mathrm{i}\omega t} = F_0\mathrm{e}^{\mathrm{i}\theta_0}\mathrm{e}^{\mathrm{i}\omega t} = F_0\mathrm{e}^{\mathrm{i}(\omega t+\theta_0)} \qquad \text{and} \qquad \mathfrak{Re}\left(G_0\mathrm{e}^{\mathrm{i}\omega t}\right) = F_0\cos(\omega t + \theta_0).$$

Let $z(t) = x(t) + \mathrm{i}y(t)$, where $x = x(t)$ and $y = y(t)$ are real valued functions of the real variable t, and consider the differential equation

$$m\frac{\mathrm{d}^2z}{\mathrm{d}t^2} + b\frac{\mathrm{d}z}{\mathrm{d}t} + kz = G_0\mathrm{e}^{\mathrm{i}\omega t}. \tag{33}$$

Taking real parts, we obtain the original differential equation (32). It follows that to find a particular integral for the original equation (32), we simply find a particular integral for (33) and take its real part. Let us try $z = G\mathrm{e}^{\mathrm{i}\omega t}$, where G is a complex valued constant to be determined. Then

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \mathrm{i}\omega G\mathrm{e}^{\mathrm{i}\omega t} \qquad \text{and} \qquad \frac{\mathrm{d}^2z}{\mathrm{d}t^2} = -\omega^2 G\mathrm{e}^{\mathrm{i}\omega t}.$$

Substituting into the left hand side of (33), we obtain

$$m\frac{\mathrm{d}^2z}{\mathrm{d}t^2} + b\frac{\mathrm{d}z}{\mathrm{d}t} + kz = (-m\omega^2G + \mathrm{i}b\omega G + kG)\mathrm{e}^{\mathrm{i}\omega t},$$

so that on equating coefficients, we obtain

$$G(k + \mathrm{i}b\omega - m\omega^2) = G_0. \tag{34}$$

Let us restrict our attention to the case of underdamped oscillations, as discussed in Example 16.11.3, and use the notation $\gamma = b/2m$ and $\omega_0 = \sqrt{k/m}$. Then (34) is equivalent to

$$G(\omega_0^2 + 2\mathrm{i}\gamma\omega - \omega^2) = \frac{G_0}{m}.$$

It follows that

$$G = \frac{G_0}{m(\omega_0^2 - \omega^2 + 2\mathrm{i}\gamma\omega)} = \frac{G_0(\omega_0^2 - \omega^2 - 2\mathrm{i}\gamma\omega)}{m(\omega_0^2 - \omega^2 + 2\mathrm{i}\gamma\omega)(\omega_0^2 - \omega^2 - 2\mathrm{i}\gamma\omega)} = \frac{G_0(\omega_0^2 - \omega^2 - 2\mathrm{i}\gamma\omega)}{m((\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2)}.$$

We can choose $\beta \in \mathbb{R}$ such that

$$\cos\beta = \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \qquad \text{and} \qquad \sin\beta = -\frac{2\gamma\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}.$$

Then

$$G = \frac{G_0(\cos\beta + \mathrm{i}\sin\beta)}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} = \frac{G_0\mathrm{e}^{\mathrm{i}\beta}}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}.$$

A particular integral for (33) is therefore given by

$$z = \frac{F_0\mathrm{e}^{\mathrm{i}\theta_0}\mathrm{e}^{\mathrm{i}\beta}\mathrm{e}^{\mathrm{i}\omega t}}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} = \frac{F_0\mathrm{e}^{\mathrm{i}(\omega t+\theta_0+\beta)}}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}.$$

A particular integral for (32) is therefore given by

$$x = \mathfrak{Re} z = \frac{F_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}\cos(\omega t + \theta_0 + \beta).$$

Problems for Chapter 16

1. Find the general solution of the following differential equations:

a) $\displaystyle\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0$

b) $\displaystyle\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 6\mathrm{e}^{-x}$

c) $\displaystyle 2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^2$

d) $\displaystyle\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} + y = 98x\mathrm{e}^{2x}$

e) $\displaystyle\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = \sin x$

f) $\displaystyle\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 9y = x^2\mathrm{e}^{3x} + 9$

g) $\displaystyle\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 75\mathrm{e}^{x}\cos x$

h) $\displaystyle\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = \cosh x$

i) $\displaystyle\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} + y = x\mathrm{e}^{2x}$

j) $\displaystyle\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 3\mathrm{e}^{2x}$

k) $\displaystyle\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} + y = 26\sin^2 x$

l) $\displaystyle\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = \cosh 2x$

m) $\displaystyle\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = \mathrm{e}^{-3x}(1 + 4x + 3x^2)$

n) $\displaystyle\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 8y = 4\mathrm{e}^{-2x}(1 + 3\cos x + 5\cos 2x)$

2. Find the solution of the following differential equations with given initial conditions:

a) $\displaystyle\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 2x$, with $y = 0$ and $\displaystyle\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$

b) $\displaystyle\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = x^2 + 3\mathrm{e}^{x}$, with $y = 0$ and $\displaystyle\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ when $x = 0$

c) $\displaystyle\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = x\mathrm{e}^{x} + 4$, with $y = 1$ and $\displaystyle\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$

d) $\displaystyle\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - y = \mathrm{e}^{x}\sin x$, with $y = 1$ and $\displaystyle\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$

e) $\displaystyle\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \mathrm{e}^{x}$, with $y = 0$ and $\displaystyle\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$

3. Find the general solutions of the following equations of motion, where the constants m, b and k are all positive, distinguishing cases if necessary:

a) $\displaystyle m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} - kx = 0$

b) $\displaystyle m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0$

4. A particle of mass m is subject to a restoring force $-kx$ and a damping force $-bv$, where x represents its displacement from equilibrium and v represents its velocity. At time $t = 0$, it is displaced a distance x_0 from equilibrium and released with zero velocity.

a) Set up a differential equation to describe the motion.

b) Solve the differential equation, distinguishing the cases of underdamping, overdamping and critical damping.

5. Repeat Problem 4 when the particle starts from equilibrium position with an initial velocity v_0.

6. An undamped harmonic oscillator is subject to an impressed force $F(t) = F_0\cos\omega t$.

a) Discuss the case when $\omega = \omega_0$, in the notation of Example 16.11.1.

b) See what happens when you try for a particular solution by starting with a solution for $\omega = \omega_0 + \epsilon$ and then passing to the limit as $\epsilon \to 0$.

c) Now try for a solution by starting with a solution for $\omega = \omega_0 + \epsilon$, fitting the initial conditions $x = 0$ and $v = v_0$ when $t = 0$, and then passing to the limit as $\epsilon \to 0$.

7. A critically damped harmonic oscillator with mass m and spring constant k is subject to an impressed force $F(t) = F_0 \cos \omega t$. Determine the displacement $x(t)$, with the initial conditions $x = x_0$ and $v = v_0$ when $t = 0$.