FIRST YEAR CALCULUS

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Chapter 17

FUNCTIONS OF TWO VARIABLES

17.1. Introduction

We have previously been concerned with functions of a single independent variable. However, there are many quantities that depend on two or more independent variables. For example, the area of a rectangle depends on its base as well as its altitude.

Real valued functions of two real variables can be represented geometrically by the z-coordinate of a point on a surface in 3-dimensional space, just as real valued functions of a single real variable can be represented geometrically by the y-coordinate of a point on a curve in 2-dimensional space.

In this chapter, we shall be concerned with functions of the type $f : D \to \mathbb{R}$, where D is a subset of $\mathbb{R} \times \mathbb{R} = \{(x, y) : x, y \in \mathbb{R}\}$. We use the convention that D is the largest set for which $f : D \to \mathbb{R}$ is a function. Throughout, $z = f(x, y)$ denotes a real valued function of two real variables x and y.

The first question we have to address is one on limits.

DEFINITION. We say that $f(x, y) \to L$ as $(x, y) \to (a, b)$, denoted by

$$\lim_{(x,y)\to(a,b)} f(x, y) = L,$$

if, given any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x, y) - L| < \epsilon$ for every $(x, y) \neq (a, b)$ satisfying $|x - a| < \delta$ and $|y - b| < \delta$.

EXAMPLE 17.1.1. Consider the function $f(x, y) = x^2 + y^3$ as $(x, y) \to (0, 0)$. We have

$$|f(x, y) - 0| = |x^2 + y^3| \le |x|^2 + |y|^3 = |x - 0|^2 + |y - 0|^3 < \epsilon$$

if $|x - 0| < (\epsilon/2)^{1/2}$ and $|y - 0| < (\epsilon/2)^{1/3}$. We may take $\delta = \min\{(\epsilon/2)^{1/2}, (\epsilon/2)^{1/3}\} = (\epsilon/2)^{1/2}$ if $\epsilon < 1$. Hence $f(x, y) \to 0$ as $(x, y) \to (0, 0)$.

Example 17.1.2. Consider the function $f(x,y) = xy$ as $(x,y) \to (1,2)$. Note that $|y| < 3$ if $|y-2| < 1$. Hence

$$|f(x,y) - 2| = |xy - 2| = |xy - y + y - 2| \le |y||x-1| + |y-2| \le 3|x-1| + |y-2| < \epsilon$$

if $|x-1| < \epsilon/4$ and $|y-2| < \min\{1, \epsilon/4\}$. We may take $\delta = \min\{1, \epsilon/4\}$. Hence $f(x,y) \to 2$ as $(x,y) \to (1,2)$.

Example 17.1.3. Consider the function

$$f(x,y) = \frac{xy}{2x^2 - y^2}.$$

If we restrict our discussion to the line $y = 0$, then $f(x,y) = 0$, so that $f(x,y)$ approaches 0 if (x,y) approaches $(0,0)$ along the line $y = 0$. If we restrict our discussion to the line $y = x$, then $f(x,y) = 1$, so that $f(x,y)$ approaches 1 if (x,y) approaches $(0,0)$ along the line $y = x$. Hence $f(x,y)$ does not have a limit as $(x,y) \to (0,0)$.

Remark. Note that the limiting value of the function has to be independent of the manner in which (x,y) approaches (a,b).

We state without proof the following results concerning the arithmetic of limits.

PROPOSITION 17A. *Suppose that $f(x,y) \to L$ and $g(x,y) \to M$ as $(x,y) \to (a,b)$. Then*
(a) $f(x,y) + g(x,y) \to L + M$ as $(x,y) \to (a,b)$;
(b) $f(x,y)g(x,y) \to LM$ as $(x,y) \to (a,b)$; and
(c) if $M \ne 0$, then $f(x,y)/g(x,y) \to L/M$ as $(x,y) \to (a,b)$.

We next define continuity in terms of limits.

Definition. We say that $f(x,y)$ is continuous at (a,b) if $f(x,y) \to f(a,b)$ as $(x,y) \to (a,b)$.

Example 17.1.4. The function $f(x,y) = x^2 + y^3$ is continuous at $(0,0)$.

Example 17.1.5. The function $f(x,y) = xy$ is continuous at $(1,2)$.

Example 17.1.6. The function

$$f(x,y) = \frac{xy}{2x^2 - y^2}$$

is not continuous at $(0,0)$. In fact, $f(0,0)$ is not even defined.

The following result is an immediate consequence of Proposition 17A and the definition of continuity.

PROPOSITION 17B. *Suppose that $f(x,y)$ and $g(x,y)$ are continuous at (a,b). Then*
(a) $f(x,y) + g(x,y)$ is continuous at (a,b);
(b) $f(x,y)g(x,y)$ is continuous at (a,b); and
(c) if $g(a,b) \ne 0$, then $f(x,y)/g(x,y)$ is continuous at (a,b).

17.2. Partial Derivatives

Consider a function $z = f(x,y)$. If y is held fixed, then z becomes a function of x alone, and its derivative, if it exists, can be found.

DEFINITION. By the partial derivative of z with respect to x, we mean the limit

$$\frac{\partial z}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h},$$

if it exists. By the partial derivative of z with respect to y, we mean the limit

$$\frac{\partial z}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h},$$

if it exists. We also write

$$\frac{\partial z}{\partial x} = f_x \qquad \text{and} \qquad \frac{\partial z}{\partial y} = f_y.$$

EXAMPLE 17.2.1. Suppose that $z = x^2y^2 + xy^4$. Keeping y fixed and differentiating with respect to x, we obtain

$$\frac{\partial z}{\partial x} = 2xy^2 + y^4.$$

Keeping x fixed and differentiating with respect to y, we obtain

$$\frac{\partial z}{\partial y} = 2x^2y + 4xy^3.$$

EXAMPLE 17.2.2. Suppose that $xy + z^2 = 16$. Keeping y fixed and differentiating with respect to x, we obtain

$$y + 2z\frac{\partial z}{\partial x} = 0.$$

Keeping x fixed and differentiating with respect to y, we obtain

$$x + 2z\frac{\partial z}{\partial y} = 0.$$

If we attempt to interpret $\partial z/\partial x$ geometrically, note first of all that the function $z = f(x,y)$ is represented by a surface in 3-dimensional space. Keeping y fixed at a value y_0 means that we are considering the intersection of the surface with a plane $y = y_0$ parallel to the xz-plane. This gives rise to a curve with z as a function of x (and the fixed y_0). The partial derivative $\partial z/\partial x$ now represents the slope of this curve.

We can define higher order derivatives. For second order derivatives, we have the following.

DEFINITION. We write

$$f_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \qquad \text{and} \qquad f_{yy} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right),$$

if the derivatives exist. We also write

$$f_{xy} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \qquad \text{and} \qquad f_{yx} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right),$$

if the derivatives exist.

EXAMPLE 17.2.3. Suppose that $z = x^2y^2 + xy^4$. We have already shown that

$$\frac{\partial z}{\partial x} = 2xy^2 + y^4 \qquad \text{and} \qquad \frac{\partial z}{\partial y} = 2x^2y + 4xy^3.$$

Differentiating again, we obtain

$$\frac{\partial^2 z}{\partial x^2} = 2y^2 \qquad \text{and} \qquad \frac{\partial^2 z}{\partial y^2} = 2x^2 + 12xy^2.$$

Also

$$\frac{\partial^2 z}{\partial y \partial x} = 4xy + 4y^3 \qquad \text{and} \qquad \frac{\partial^2 z}{\partial x \partial y} = 4xy + 4y^3.$$

REMARKS. (1) Note in Example 17.2.3 that

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}.$$

This is not true in general. However, equality will hold if all the derivatives involved are continuous. The proof of this is rather complicated.

(2) Note that when we differentiate $z = f(x, y)$ with respect to x to obtain f_x, we keep y fixed. It follows that we can use the sum, product and quotient rules to carry out the differentiation with respect to x. A similar remark applies when we differentiate $z = f(x, y)$ with respect to y or when we attempt to obtain higher order derivatives.

17.3. The Differential

Consider a function $z = f(x, y)$. Let us now give x an increment Δx and give y an increment Δy, and suppose that this results in an increment Δz for z. Then

$$z + \Delta z = f(x + \Delta x, y + \Delta y),$$

so that

$$\begin{aligned} \Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y) \\ &= f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) + f(x, y + \Delta y) - f(x, y) \\ &= \frac{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)}{\Delta x} \Delta x + \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \Delta y \\ &= \left(\frac{\partial z}{\partial x} + \epsilon_1 \right) \Delta x + \left(\frac{\partial z}{\partial y} + \epsilon_2 \right) \Delta y, \end{aligned}$$

where ϵ_1 and ϵ_2 both approach 0 when Δx and Δy approach 0. Hence

$$\Delta z = \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y, \tag{1}$$

where $\epsilon_1 \Delta x + \epsilon_2 \Delta y$ is negligible. If we imagine quantities $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$ instead of $\Delta x, \Delta y, \Delta z$, then

$$\mathrm{d}z = \frac{\partial z}{\partial x} \, \mathrm{d}x + \frac{\partial z}{\partial y} \, \mathrm{d}y.$$

DEFINITION. The imagined quantities $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$ are called differentials.

EXAMPLE 17.3.1. Suppose that $z = x^2y^5 + 3xy - 5xy^3$. Then

$$\mathrm{d}z = (2xy^5 + 3y - 5y^3)\,\mathrm{d}x + (5x^2y^4 + 3x - 15xy^2)\,\mathrm{d}y.$$

EXAMPLE 17.3.2. Suppose that $z = (x^3 + y^2)^4$. Then

$$\mathrm{d}z = 12x^2(x^3 + y^2)^3\,\mathrm{d}x + 8y(x^3 + y^2)^3\,\mathrm{d}y.$$

17.4. Directional Derivatives

Suppose that $l = (\cos\theta, \sin\theta)$ is a unit vector on the xy-plane. Let us move by an imagined distance $\mathrm{d}l$ from the point (x, y) in the direction of l. If $\mathrm{d}x$ and $\mathrm{d}y$ represent the changes in x and y respectively, then we have

$$\frac{\mathrm{d}x}{\mathrm{d}l} = \cos\theta \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}l} = \sin\theta.$$

DEFINITION. The directional derivative in the direction of l is defined by

$$\frac{\mathrm{d}z}{\mathrm{d}l} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}l} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}l} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) \cdot (\cos\theta, \sin\theta),$$

where $\cdot$ denotes the scalar product in $\mathbb{R}^2$.

REMARK. In the direction of the positive x-axis, we have $\theta = 0$, so that the directional derivative is the partial derivative with respect to x; in other words,

$$\frac{\mathrm{d}z}{\mathrm{d}l} = \frac{\partial z}{\partial x}.$$

In the direction of the positive y-axis, we have $\theta = \pi/2$, so that the directional derivative is the partial derivative with respect to y; in other words,

$$\frac{\mathrm{d}z}{\mathrm{d}l} = \frac{\partial z}{\partial y}.$$

EXAMPLE 17.4.1. Suppose that $z = x^2 + y^2 + 1$. If $\theta = \pi/4$, then

$$\frac{\mathrm{d}z}{\mathrm{d}l} = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) = \sqrt{2}(x + y).$$

If $\theta = \pi/2$, then

$$\frac{\mathrm{d}z}{\mathrm{d}l} = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) \cdot (0, 1) = \frac{\partial z}{\partial y} = 2y.$$

17.5. The Total Derivative

Consider a function $z = f(x,y)$, where x and y are real valued functions of a real variable t. To calculate $\mathrm{d}z/\mathrm{d}t$, we can first express z in terms of t, and then differentiate by the usual rules. However, note that if t is given an increment Δt, resulting in increments $\Delta x, \Delta y, \Delta z$ for x, y, z respectively, then it follows from (1) that

$$\frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t} + \epsilon_1 \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t},$$

where ϵ_1 and ϵ_2 are negligible when Δt is small. We now imagine quantities $\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z$ instead of $\Delta t, \Delta x, \Delta y, \Delta z$.

PROPOSITION 17C. *Suppose that $z = f(x,y)$, where x and y are real valued functions of a real variable t. Then*

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}, \qquad (2)$$

provided that all derivatives exist.

DEFINITION. The derivative (2) is called the total derivative of z with respect to t.

EXAMPLE 17.5.1. Suppose that $z = x^2 + y^2 + 1$, where $x = t^3$ and $y = t^2 + 1$. Then $z = t^6 + (t^2+1)^2 + 1$, so that

$$\frac{\mathrm{d}z}{\mathrm{d}t} = 6t^5 + 4t(t^2+1).$$

If we use Proposition 17C, then

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = 6xt^2 + 4yt = 6t^5 + 4t(t^2+1).$$

EXAMPLE 17.5.2. Suppose that $z = x^3 + x\mathrm{e}^y$, where $x = \sin t$ and $y = \log t$. Then

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = (3x^2 + \mathrm{e}^y)\cos t + x\mathrm{e}^y t^{-1} = (3\sin^2 t + t)\cos t + \sin t.$$

17.6. Change of Variables

Consider a function $z = f(x,y)$, where $x = x(s,t)$ and $y = y(s,t)$ are real valued functions of two real variables s and t. Then $z = g(s,t)$ is a function of s and t.

Suppose that we keep s fixed and differentiate z with respect to t. Then it follows from (2) that

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Note that since we keep s fixed, the differentiation with respect to t results in partial derivatives with respect to t. Similarly, if we keep t fixed and differentiate z with respect to s, then we have

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}.$$

We summarize these observations as follows.

PROPOSITION 17D. *Suppose that $z = f(x, y)$, where x and y are real valued functions of two real variables s and t. Then*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \textit{and} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

REMARK. Suppose that s and t are independent variables. Then

$$\begin{aligned}
\mathrm{d}z &= \frac{\partial z}{\partial s}\,\mathrm{d}s + \frac{\partial z}{\partial t}\,\mathrm{d}t \\
&= \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\right)\mathrm{d}s + \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right)\mathrm{d}t \\
&= \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial s}\,\mathrm{d}s + \frac{\partial x}{\partial t}\,\mathrm{d}t\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial s}\,\mathrm{d}s + \frac{\partial y}{\partial t}\,\mathrm{d}t\right) \\
&= \frac{\partial z}{\partial x}\,\mathrm{d}x + \frac{\partial z}{\partial y}\,\mathrm{d}y.
\end{aligned}$$

EXAMPLE 17.6.1. The transformation from rectangular coordinates to polar coordinates is given by $x = r\cos\theta$ and $y = r\sin\theta$. Then

$$\begin{aligned}
\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta, \\
\frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial z}{\partial x}r\sin\theta + \frac{\partial z}{\partial y}r\cos\theta.
\end{aligned}$$

Combining these two equations, we find that

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\cos\theta - \frac{\partial z}{\partial \theta}\frac{\sin\theta}{r} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\sin\theta + \frac{\partial z}{\partial \theta}\frac{\cos\theta}{r}.$$

17.7. Tangent Planes and Normals

Consider the surface in 3-dimensional space that represents a function $z = f(x, y)$. Let (x, y, z) be a point on this surface, and let $(x + \Delta x, y + \Delta y, z + \Delta z)$ be a neighbouring point on the surface. Then if $\Delta x, \Delta y, \Delta z$ are very small, then the line joining (x, y, z) and $(x + \Delta x, y + \Delta y, z + \Delta z)$ is almost on the tangent plane to the surface at (x, y, z). Note that a vector in the direction of this line is given by $(\Delta x, \Delta y, \Delta z)$. To find a normal to the surface at (x, y, z), we therefore need to find a vector which will be perpendicular to $(\Delta x, \Delta y, \Delta z)$ whenever $\Delta x, \Delta y, \Delta z \to 0$. In other words, we need to find a vector $\mathbf{v} \in \mathbb{R}^3$ such that

$$\mathbf{v} \cdot (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z) = 0. \tag{3}$$

Since

$$\mathrm{d}z = \frac{\partial z}{\partial x}\,\mathrm{d}x + \frac{\partial z}{\partial y}\,\mathrm{d}y,$$

clearly the vector

$$\mathbf{v} = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$$

satisfies (3). We summarize these observations as follows.

PROPOSITION 17E. *Suppose that (x_0, y_0, z_0) is a point on the surface in 3-dimensional space that represents a function $z = f(x, y)$.*

(a) A normal vector to the surface at (x_0, y_0, z_0) is given by

$$\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)_{(x_0,y_0,z_0)}.$$

(b) The equation of the tangent plane to the surface at (x_0, y_0, z_0) is given by

$$(x - x_0, y - y_0, z - z_0) \cdot \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)_{(x_0,y_0,z_0)} = 0,$$

where $\cdot$ denotes the scalar product in $\mathbb{R}^3$.

(c) The equation of the line normal to the surface at (x_0, y_0, z_0) is given by

$$(x - x_0, y - y_0, z - z_0) = t\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)_{(x_0,y_0,z_0)}.$$

EXAMPLE 17.7.1. Consider the ellipsoid $x^2 + 2y^2 + 4z^2 = 26$ at the point $(2, -3, -1)$. Then

$$2x + 8z\frac{\partial z}{\partial x} = 0 \qquad \text{and} \qquad 4y + 8z\frac{\partial z}{\partial y} = 0,$$

so that

$$\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)_{(2,-3,-1)} = \left(-\frac{x}{4z}, -\frac{y}{2z}, -1\right)_{(2,-3,-1)} = \left(\frac{1}{2}, -\frac{3}{2}, -1\right).$$

It follows that the equation of the tangent plane at $(2, -3, -1)$ is given by

$$(x - 2, y + 3, z + 1) \cdot \left(\frac{1}{2}, -\frac{3}{2}, -1\right) = 0;$$

in other words, $x - 3y - 2z = 13$. The equation of the normal at $(2, -3, -1)$ is given by

$$(x - 2, y + 3, z + 1) = t\left(\frac{1}{2}, -\frac{3}{2}, -1\right);$$

in other words,

$$x - 2 = -\frac{y + 3}{3} = -\frac{z + 1}{2}.$$

17.8. Stationary Points

Suppose that a function $z = f(x, y)$ has continuous second partial derivatives. When the function has a maximum or minimum point, the tangent plane is then horizontal, so that a normal vector is given by $(0, 0, 1)$. Note, however, that the normal vector is given by

$$\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right).$$

DEFINITION. We say that $f(x,y)$ has a stationary point at (x,y) if

$$\frac{\partial z}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial z}{\partial y} = 0.$$

EXAMPLE 17.8.1. Consider the function $z = x^2 - y^2$. Then

$$\frac{\partial z}{\partial x} = 2x \qquad \text{and} \qquad \frac{\partial z}{\partial y} = -2y,$$

so that $(0,0)$ is a stationary point. Suppose that we intersect the surface with the plane $x = 0$. Then $z = -y^2$, and there is a maximum at $y = 0$. Suppose that we intersect the surface with the plane $y = 0$. Then $z = x^2$, and there is a minimum at $x = 0$. It follows that $f(x,y)$ has neither a maximum nor a minimum at the point $(0,0)$. In fact, it has a saddle point.

We state without proof the following result.

PROPOSITION 17F. *Suppose that a function $z = f(x,y)$ has continuous second partial derivatives. Suppose further that*

$$\frac{\partial z}{\partial x} = 0 \qquad \textit{and} \qquad \frac{\partial z}{\partial y} = 0$$

at the point (x_0, y_0). Write

$$\Delta = \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2.$$

(a) If $\Delta > 0$ and $\partial^2 z/\partial x^2 < 0$ at (x_0, y_0), then $f(x,y)$ has a maximum at (x_0, y_0).
(b) If $\Delta > 0$ and $\partial^2 z/\partial x^2 > 0$ at (x_0, y_0), then $f(x,y)$ has a minimum at (x_0, y_0).
(c) If $\Delta < 0$, then $f(x,y)$ has a saddle point at (x_0, y_0).

EXAMPLE 17.8.2. Suppose that $z = x^2 + 4y^2 - 2x + 8y - 1$. Then

$$\frac{\partial z}{\partial x} = 2x - 2 \qquad \text{and} \qquad \frac{\partial z}{\partial y} = 8y + 8,$$

so that there is a stationary point at $(1,-1)$. Now

$$\frac{\partial^2 z}{\partial x^2} = 2 \qquad \text{and} \qquad \frac{\partial^2 z}{\partial y^2} = 8 \qquad \text{and} \qquad \frac{\partial^2 z}{\partial x \partial y} = 0,$$

so that $\Delta > 0$. Hence there is a minimum at $(1,-1)$, with $z = -6$.

EXAMPLE 17.8.3. Suppose that $z = x^3 - y^3 - 3xy + 4$. Then

$$\frac{\partial z}{\partial x} = 3x^2 - 3y \qquad \text{and} \qquad \frac{\partial z}{\partial y} = -3y^2 - 3x.$$

For stationary points, we need

$$\begin{aligned} 3x^2 - 3y &= 0, \\ -3y^2 - 3x &= 0, \end{aligned}$$

so that there are two stationary points, at $(0,0)$ and $(-1,1)$. Now

$$\frac{\partial^2 z}{\partial x^2} = 6x \qquad \text{and} \qquad \frac{\partial^2 z}{\partial y^2} = -6y \qquad \text{and} \qquad \frac{\partial^2 z}{\partial x \partial y} = -3.$$

At $(0,0)$, we have

$$\frac{\partial^2 z}{\partial x^2} = 0 \qquad \text{and} \qquad \frac{\partial^2 z}{\partial y^2} = 0 \qquad \text{and} \qquad \frac{\partial^2 z}{\partial x \partial y} = -3,$$

so that $\Delta = -9 < 0$. Hence there is a saddle point at $(0,0)$. At $(-1,1)$, we have

$$\frac{\partial^2 z}{\partial x^2} = -6 \qquad \text{and} \qquad \frac{\partial^2 z}{\partial y^2} = -6 \qquad \text{and} \qquad \frac{\partial^2 z}{\partial x \partial y} = -3,$$

so that $\Delta = 27 > 0$. Hence there is a maximum at $(-1,1)$, with $z = 5$.

EXAMPLE 17.8.4. Suppose that $z = x^3 + y^3 - 3x - 12y + 4$. Then

$$\frac{\partial z}{\partial x} = 3x^2 - 3 \qquad \text{and} \qquad \frac{\partial z}{\partial y} = 3y^2 - 12.$$

For stationary points, we need $3x^2 - 3 = 0$ and $3y^2 - 12 = 0$, so that there are four stationary points, at $(1,2)$, $(1,-2)$, $(-1,2)$ and $(-1,-2)$. Now

$$\frac{\partial^2 z}{\partial x^2} = 6x \qquad \text{and} \qquad \frac{\partial^2 z}{\partial y^2} = 6y \qquad \text{and} \qquad \frac{\partial^2 z}{\partial x \partial y} = 0.$$

At $(1,2)$, we have

$$\frac{\partial^2 z}{\partial x^2} = 6 \qquad \text{and} \qquad \frac{\partial^2 z}{\partial y^2} = 12 \qquad \text{and} \qquad \frac{\partial^2 z}{\partial x \partial y} = 0,$$

so that $\Delta = 72 > 0$. Hence there is a minimum at $(1,2)$, with $z = -14$. At $(-1,-2)$, we have

$$\frac{\partial^2 z}{\partial x^2} = -6 \qquad \text{and} \qquad \frac{\partial^2 z}{\partial y^2} = -12 \qquad \text{and} \qquad \frac{\partial^2 z}{\partial x \partial y} = 0,$$

so that $\Delta = 72 > 0$. Hence there is a maximum at $(-1,-2)$, with $z = 22$. It can be checked that $\Delta < 0$ at $(1,-2)$ and $(-1,2)$, so that these two stationary points are saddle points.

17.9. An Application to Ordinary Differential Equations

We conclude this chapter using partial derivatives to study a first order ordinary differential equation of the type

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{P(x,y)}{Q(x,y)},$$

where $P(x,y)$ and $Q(x,y)$ are two given functions. In other words, we consider equations of the form

$$Q(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} + P(x,y) = 0. \tag{4}$$

For certain forms of $P(x,y)$ and $Q(x,y)$, it may be possible to write the left hand side of (4) as the total differential coefficient of some function $u(x,y)$, where

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial u}{\partial x}. \tag{5}$$

Let us compare (4) and (5). If

$$P(x,y) = \frac{\partial u}{\partial x} \qquad \text{and} \qquad Q(x,y) = \frac{\partial u}{\partial y}, \tag{6}$$

then (4) can be written in the form

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 0,$$

giving the solution

$$u(x,y) = C. \tag{7}$$

Differentiating (6) and assuming that $P(x,y)$ and $Q(x,y)$ have continuous first derivatives, we have

$$\frac{\partial P}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial Q}{\partial x}. \tag{8}$$

This condition is a necessary and sufficient condition for the equation (4) to be expressed as an exact or total differentiable coefficient. We say that the equation (4) is exact if the condition (8) is satisfied. The solution is then given by (7).

Example 17.9.1. Suppose that

$$(8 - x^2)y\frac{\mathrm{d}y}{\mathrm{d}x} + x(1 - y^2) = 0.$$

Then $P(x,y) = x(1 - y^2)$ and $Q(x,y) = (8 - x^2)y$. Since

$$\frac{\partial P}{\partial y} = -2xy = \frac{\partial Q}{\partial x},$$

the equation is exact. It follows that the solution is of the form $u(x,y) = C$, where

$$\frac{\partial u}{\partial x} = x(1 - y^2) \qquad \text{and} \qquad \frac{\partial u}{\partial y} = (8 - x^2)y.$$

Integrating, we have

$$u(x,y) = \frac{1}{2}x^2(1 - y^2) + f(y) \qquad \text{and} \qquad u(x,y) = \frac{1}{2}(8 - x^2)y^2 + g(x)$$

for some functions $f(y)$ and $g(x)$. Since we must have

$$\frac{1}{2}x^2(1 - y^2) + f(y) = \frac{1}{2}(8 - x^2)y^2 + g(x),$$

we can take

$$f(y) = 4y^2 \qquad \text{and} \qquad g(x) = \frac{1}{2}x^2,$$

so that

$$u(x,y) = \frac{1}{2}x^2 - \frac{1}{2}x^2y^2 + 4y^2.$$

Hence the solution is $x^2(1 - y^2) + 8y^2 = A$ for some constant A.

EXAMPLE 17.9.2. Suppose that

$$(2x\log x)\frac{\mathrm{d}y}{\mathrm{d}x}+y=0.$$

Then $P(x,y)=y$ and $Q(x,y)=2x\log x$. Since

$$\frac{\partial P}{\partial y}=1\neq 2\log x+2=\frac{\partial Q}{\partial x},$$

the equation is not exact. Let us multiply through by y/x, and consider the same equation in the form

$$(2y\log x)\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{y^2}{x}=0.$$

Now $P(x,y)=y^2/x$ and $Q(x,y)=2y\log x$. Since

$$\frac{\partial P}{\partial y}=\frac{2y}{x}=\frac{\partial Q}{\partial x},$$

the equation is now exact. It follows that the solution is of the form $u(x,y)=C$, where

$$\frac{\partial u}{\partial x}=\frac{y^2}{x}\qquad\text{and}\qquad\frac{\partial u}{\partial y}=2y\log x.$$

Integrating, we have

$$u(x,y)=y^2\log x+f(y)\qquad\text{and}\qquad u(x,y)=y^2\log x+g(x)$$

for some functions $f(y)$ and $g(x)$. Clearly we can take $f(y)=0$ and $g(x)=0$, so that $u(x,y)=y^2\log x$. Hence the solution is $y^2\log x=C$ for some constant C.

Example 17.9.2 suggests the following technique. If the equation (4) is not exact, then we multiply the equation by an integrating factor $\mu(x,y)$ and consider the equation

$$\mu(x,y)Q(x,y)\frac{\mathrm{d}y}{\mathrm{d}x}+\mu(x,y)P(x,y)=0. \tag{9}$$

Needless to say, we attempt to choose $\mu(x,y)$ in order to make the new equation (9) exact. We therefore must have

$$\frac{\partial}{\partial x}(\mu Q)=\frac{\partial}{\partial y}(\mu P); \tag{10}$$

in other words,

$$\mu\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)+Q\frac{\partial \mu}{\partial x}-P\frac{\partial \mu}{\partial y}=0. \tag{11}$$

There is no need to remember (11), as it is easily deduced from (10), the analogue of (8). Unfortunately, the equations (10) and (11) are not easy to solve except when P and Q have very simple forms.

EXAMPLE 17.9.3. Suppose that

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0.$$

Then $P(x,y) = -y$ and $Q(x,y) = x$. It is easy to see that the equation is not exact. By (11), any integrating factor must satisfy

$$2\mu + x\frac{\partial\mu}{\partial x} + y\frac{\partial\mu}{\partial y} = 0.$$

We can take $\mu(x,y)$ to be

$$\pm\frac{1}{x^2} \qquad \text{or} \qquad \pm\frac{1}{y^2} \qquad \text{or} \qquad \pm\frac{1}{xy} \qquad \text{or} \qquad \pm\frac{1}{x^2+y^2} \qquad \text{or} \qquad \pm\frac{1}{x^2-y^2}.$$

Let us choose $\mu(x,y) = 1/x^2$. Then the equation becomes

$$\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x^2} = 0,$$

with solution $u(x,y) = C$, where

$$\frac{\partial u}{\partial x} = -\frac{y}{x^2} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = \frac{1}{x}.$$

Integrating, we have

$$u(x,y) = \frac{y}{x} + f(y) \qquad \text{and} \qquad u(x,y) = \frac{y}{x} + g(x)$$

for some functions $f(y)$ and $g(x)$. Clearly we can take $f(y) = 0$ and $g(x) = 0$, so that $u(x,y) = y/x$. Hence the solution is $y/x = C$ for some constant C.

Problems for Chapter 17

1. For each of the following, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$:
   a) $z = x^3 + 7x^6y^2 + 8x$
   b) $z = x\sin y + y\cos x$
   c) $x^2 + y^2 - z^2 = 1$
   d) $xyz = 1$

2. If $z = \dfrac{x^4 - y^4}{xy}$, verify that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2z$.

3. For each of the following, find $\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial y^2}$ and $\dfrac{\partial^2 z}{\partial x \partial y}$:
   a) $z = x^3 + 7x^6y^2 + 8x$
   b) $z = x\sin y + y\cos x$
   c) $xy + yz + xz = 1$

4. Verify that $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y}$ if $z = \sin(3x + 2y)$.

5. Verify that $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = 0$ if $z = \cos(x + y) + \cos(x - y)$.

6. Find the differential of each of the following functions:
   a) $z = x^3y + x^2y^2 + 3$
   b) $z = \log(xy)$
   c) $z = \cosh(x^2 + y^2)$

7. For each of the following functions, find the derivative in the direction of the vector indicated (note that the vectors are not unit vectors):
   a) $z = x^3y + x^2y^2 + 3$ and $(1, 2)$
   b) $z = \log(xy)$ and $(-1, 1)$
   c) $z = \cosh(x^2 + y^2)$ and $(3, 4)$

8. For each of the following functions, find $\dfrac{\mathrm{d}z}{\mathrm{d}t}$:
   a) $z = x^2 + 4y^2$, where $x = \sin t$ and $y = \cos t$
   b) $z = \mathrm{e}^x \sin y$, where $x = \log t$ and $y = t^2$

9. For each of the following functions, find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$:
   a) $z = \log\sqrt{x^2 + y^2}$, where $x = s\mathrm{e}^t$ and $y = s\mathrm{e}^{-t}$
   b) $z = x^2y^3$, where $x = s^{-1}\sin t$ and $y = st^{-1}$
   c) $z = \mathrm{e}^{y/x}$, where $x = s\cos t$ and $y = s\sin t$

10. If $z = f(x - y)$, show that $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$.

11. For each of the following surfaces, find the equations of the tangent plane and normal line at the point indicated:
    a) $x^2 + y^2 + z^2 = 14$ at $(-2, 1, 3)$
    b) $x^2 + 4y^2 = 2z$ at $(2, 1, 4)$
    c) $x^2 + 3y^2 - 4z^2 + 3x - 2y + 10z - 42 = 0$ at $(4, 2, 1)$

12. Show that the equation of the tangent plane to the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ at the point (x_0, y_0, z_0) is given by $\dfrac{x_0x}{a^2} + \dfrac{y_0y}{b^2} + \dfrac{z_0z}{c^2} = 1$.

13. Show that the sphere $x^2 + y^2 + z^2 = 2a^2$ and the cylinder $yz = a^2$ have the same tangent plane at the point $(0, a, a)$.

14. For each of the following functions, find all stationary points and determine, if possible, their nature:
   a) $z = x^2 - y^2 + 6x - 10y + 2$
   b) $z = x^2 + 4xy + y^2 - 6y + 1$
   c) $z = \mathrm{e}^{-(x^2+y^2)}$
   d) $z = (x^2 + y^2)^2 - 2(x^2 - y^2)$
   e) $z = (x + 2y + 2)/(x^2 + y^2 + 1)$

15. Find by the use of derivatives the shortest distance from the origin to the plane $x + y + z = a$.

16. For each of the following, show that the differential equation is exact, and find its general solution:
   a) $2xy\dfrac{\mathrm{d}y}{\mathrm{d}x} + 3x^2 + y^2 = 0$
   b) $\sinh x \sinh y\dfrac{\mathrm{d}y}{\mathrm{d}x} + \cosh x \cosh y = 0$
   c) $x \cos x\dfrac{\mathrm{d}y}{\mathrm{d}x} + (\cos x - x \sin x)y = 0$
   d) $\mathrm{e}^y \sin x\dfrac{\mathrm{d}y}{\mathrm{d}x} + (1 + \mathrm{e}^y) \cos x = 0$

17. Consider the differential equation

$$(x^3 - 2xy)\frac{\mathrm{d}y}{\mathrm{d}x} + x + 2y^2 = 0.$$

   a) Show that the equation can be made exact by multiplying the equation by a suitable power of the independent variable x.
   b) Find the general solution of the equation.