FIRST YEAR CALCULUS

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Chapter 19

SEQUENCES

19.1. Introduction

A sequence (of numbers) is a set of numbers occurring in order. In simple cases, a sequence is defined by an explicit formula giving the n-th term x_n in terms of n. We shall simply refer to the sequence x_n. For example, $x_n = 1/n$ represents the sequence

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots.$$

We shall only be concerned with the case when all the terms of a sequence are real, so that throughout this chapter, x_n represents a real sequence.

It is not necessary to start the sequence with x_1. However, the set of all natural numbers is a convenient tool to indicate the order with which the numbers occur.

Remark. Formally, a real sequence is a function of the form $f : \mathbb{N} \to \mathbb{R}$, where for every $n \in \mathbb{N}$, we write $f(n) = x_n$.

Let us now investigate how a sequence may behave. We begin by looking at three examples.

Example 19.1.1. Consider the real sequence $x_n = 1/n$. We are interested in the behaviour of x_n as n gets large. It is easy to see that as n gets larger, then x_n gets smaller. In fact, as n gets very large, then x_n gets very close to 0. In this case, we say that $x_n \to 0$ as $n \to \infty$.

Example 19.1.2. Consider the real sequence $x_n = n^2$. It is easy to see that as n gets larger, then x_n also gets larger. In fact, x_n can get arbitrarily large, as long as n is large enough. In this case, we say that $x_n \to \infty$ as $n \to \infty$.

Example 19.1.3. Consider the real sequence $x_n = (-1)^n$. It is easy to see that as n gets larger, then x_n alternates between the values ± 1, and does not get close to any real number or become infinite.

Definition. We say that a real sequence x_n converges to a finite limit $x \in \mathbb{R}$, denoted by $x_n \to x$ as $n \to \infty$ or by

$$\lim_{x\to\infty} x_n = x,$$

if, given any $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that $|x_n - x| < \epsilon$ whenever $n > N$.

Note that the quantity $|x_n - x|$ measures the difference between x_n and its intended limit x. The definition thus says that this difference can be made as small as we like, provided that n is large enough. Note here that the choice of the real number N may well depend on the choice of the number ϵ.

Definition. We say that a real sequence x_n is convergent if it converges to some finite limit x as $n \to \infty$. Otherwise, we say that x_n is divergent.

Example 19.1.4. Consider the sequence $x_n = 1/n$. Then $x_n \to 0$ as $n \to \infty$, since

$$|x_n - 0| = 1/n < \epsilon \quad \text{whenever } n > N = \frac{1}{\epsilon}.$$

Example 19.1.5. Consider the sequence $x_n = 1/n^2$. Then $x_n \to 0$ as $n \to \infty$, since

$$|x_n - 0| = 1/n^2 < \epsilon \quad \text{whenever } n > N = \sqrt{\frac{1}{\epsilon}}.$$

Example 19.1.6. Consider the sequence $x_n = (n+2)/n$. Then $x_n \to 1$ as $n \to \infty$, since

$$\left|\frac{n+2}{n} - 1\right| = \frac{2}{n} < \epsilon \quad \text{whenever } n > N = \frac{2}{\epsilon}.$$

Example 19.1.7. Consider the sequence $x_n = \sqrt{(n+1)/n}$. Then $x_n \to 1$ as $n \to \infty$, since

$$\left|\sqrt{\frac{n+1}{n}} - 1\right| = \frac{\frac{n+1}{n} - 1}{\sqrt{\frac{n+1}{n}} + 1} < \frac{1}{2n} < \epsilon \quad \text{whenever } n > N = \frac{1}{2\epsilon}.$$

Example 19.1.8. Consider the sequence $x_n = (2n+3)/(3n+4)$. Then $x_n \to 2/3$ as $n \to \infty$, since

$$\left|\frac{2n+3}{3n+4} - \frac{2}{3}\right| = \frac{1}{3(3n+4)} < \frac{1}{9n} < \epsilon \quad \text{whenever } n > N = \frac{1}{9\epsilon}.$$

Remark. Note that the inequality $|x_n - x| < \epsilon$ is equivalent to the inequalities $x - \epsilon < x_n < x + \epsilon$. Note also that the convergence of a sequence is not affected by the initial terms.

A simple and immediate consequence of our definition of convergence is the following result which we shall prove in Section 19.4.

PROPOSITION 19A. *The limit of a convergent real sequence is unique.*

Definition. A real sequence x_n is said to be bounded if there exists a number $M \in \mathbb{R}$ such that $|x_n| \le M$ for every $n \in \mathbb{N}$.

Example 19.1.9. The sequence $x_n = 1/n$ is bounded. Clearly $|x_n| \le 1$ for every $n \in \mathbb{N}$.

Example 19.1.10. The sequence $x_n = 1/n^2$ is bounded. Clearly $|x_n| \le 1$ for every $n \in \mathbb{N}$.

Example 19.1.11. The sequence $x_n = (n+2)/n$ is bounded. Clearly

$$|x_n| = \frac{n+2}{n} = 1 + \frac{2}{n} \le 3 \quad \text{for every } n \in \mathbb{N}.$$

Example 19.1.12. The sequence $x_n = \sqrt{(n+1)/n}$ is bounded. Clearly

$$|x_n| = \sqrt{\frac{n+1}{n}} = \sqrt{1 + \frac{1}{n}} \le \sqrt{2} \quad \text{for every } n \in \mathbb{N}.$$

Example 19.1.13. The sequence $x_n = (2n+3)/(3n+4)$ is bounded. Clearly

$$|x_n| = \frac{2n+3}{3n+4} \le \frac{2n+3}{3n} = \frac{2}{3} + \frac{1}{n} \le \frac{5}{3} \quad \text{for every } n \in \mathbb{N}.$$

Note that the bounded sequences in Examples 19.1.9–19.1.13 are also the convergent sequences in Examples 19.1.4–19.1.8 respectively. These are examples which illustrate the fact that convergence implies boundedness. More precisely, we have the following result which we shall prove in Section 19.4.

PROPOSITION 19B. *A convergent real sequence is bounded.*

The next example shows that a bounded real sequence is not necessarily convergent.

Example 19.1.14. The sequence $x_n = (-1)^n$ is bounded. Clearly $|x_n| \le 1$ for every $n \in \mathbb{N}$. We now show that this sequence is not convergent. Let x be any given real number. We shall show that the sequence x_n does not converge to x. Note first of all that for every $n \in \mathbb{N}$, we have $|x_{n+1} - x_n| = 2$. We next use the triangle inequality, that for any $\alpha, \beta \in \mathbb{R}$, we have $|\alpha + \beta| \le |\alpha| + |\beta|$. By taking $\alpha = x_{n+1} - x$ and $\beta = x - x_n$, we have

$$2 = |x_{n+1} - x_n| = |x_{n+1} - x + x - x_n| \le |x_{n+1} - x| + |x - x_n| = |x_{n+1} - x| + |x_n - x|.$$

It follows that for every $n \in \mathbb{N}$, at least one of the two inequalities $|x_{n+1} - x| \ge 1$ and $|x - x_n| \ge 1$ must hold. This clearly shows that the condition for convergence cannot be satisfied with $\epsilon = 1$.

The next result shows that we can do arithmetic on limits. See Section 19.4 for proofs.

PROPOSITION 19C. *Suppose that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Then*
(a) $x_n + y_n \to x + y$ as $n \to \infty$;
(b) $x_n y_n \to xy$ as $n \to \infty$; and
(c) if $y \ne 0$, then $x_n/y_n \to x/y$ as $n \to \infty$.

Remark. Let $y_n = 1/n$ and $z_n = (-1)^n$. Then $y_n \to 0$ as $n \to \infty$, but z_n does not converge as $n \to \infty$. On the other hand, it is easy to check that $x_n = y_n z_n \to 0$ as $n \to \infty$. Note now that $z_n = x_n/y_n$, but since $y_n \to 0$ as $n \to \infty$, we cannot use Proposition 19C(c).

Definition. We say that $x_n \to \infty$ as $n \to \infty$ if, for every $E > 0$, there exists $N \in \mathbb{R}$ such that $|x_n| > E$ whenever $n > N$. In this case, we say that the sequence x_n diverges to ∞ as $n \to \infty$.

REMARKS. (1) It can be shown that $x_n \to \infty$ as $n \to \infty$ if and only if $1/x_n \to 0$ as $n \to \infty$.

(2) Note that Proposition 19C does not apply in the case when a sequence diverges to ∞.

EXAMPLE 19.1.15. The sequences $x_n = n$, $x_n = n^2$ and $x_n = (-1)^n n$ all satisfy $x_n \to \infty$ as $n \to \infty$.

EXAMPLE 19.1.16. Suppose that x_n is a sequence of positive terms such that $x_n \to 0$ as $n \to \infty$. For every fixed $m \in \mathbb{N}$, we have $x_n^m \to 0$ as $n \to \infty$, in view of Proposition 19C(b). For every negative integer m, we have $x_n^m \to \infty$ as $n \to \infty$, noting that $x_n > 0$ for every $n \in \mathbb{N}$. How about $m = 0$?

19.2. Special Results for Real Sequences

Note that our discussion up to this point can be extended to sequences of complex numbers. However, real sequences are particularly interesting since the real numbers are ordered (unlike the complex numbers). This enables us to establish special results for convergence which apply only to real sequences. Detailed proofs will be given in Section 19.4.

We begin with a simple example. Imagine that you have a ham sandwich, and you do the most disgusting thing of squeezing the two slices of bread together. Where does the ham go?

PROPOSITION 19D. (SQUEEZING PRINCIPLE) *Suppose that $x_n \to x$ and $y_n \to x$ as $n \to \infty$. Suppose further that $x_n \le a_n \le y_n$ for every $n \in \mathbb{N}$. Then $a_n \to x$ as $n \to \infty$.*

EXAMPLE 19.2.1. Consider the sequence

$$a_n = \frac{4n+3}{4n^2+3n+1}.$$

Then

$$\frac{1}{2n} = \frac{4n}{8n^2} < \frac{4n+3}{4n^2+3n+1} < \frac{4n+3+n^{-1}}{4n^2+3n+1} = \frac{1}{n}.$$

Writing

$$x_n = \frac{1}{2n} \qquad \text{and} \qquad y_n = \frac{1}{n},$$

we have that $x_n \to 0$ and $y_n \to 0$ as $n \to \infty$. Hence $a_n \to 0$ as $n \to \infty$.

EXAMPLE 19.2.2. Consider the sequence $a_n = n^{-1} \cos n$. Writing $x_n = -1/n$ and $y_n = 1/n$, we have $x_n \le a_n \le y_n$ for every $n \in \mathbb{N}$. Since $x_n \to 0$ and $y_n \to 0$ as $n \to \infty$, we have $a_n \to 0$ as $n \to \infty$.

EXAMPLE 19.2.3. It is important that x_n and y_n converge to the same limit. For example, if $x_n = -1$ and $y_n = 1$ for every $n \in \mathbb{N}$, then both x_n and y_n converge as $n \to \infty$. Let $a_n = (-1)^n$. Then $x_n \le a_n \le y_n$ for every $n \in \mathbb{N}$. Note from Example 19.1.14 that a_n does not converge as $n \to \infty$. In this case, the hypotheses of Proposition 19D are not satisfied. Note that x_n and y_n converge to different limits, so no "squeezing" occurs.

EXAMPLE 19.2.4. Consider the sequence $x_n = a^n$, where $a \in \mathbb{R}$. There are various cases:

- If $a = 1$, then $x_n = 1$ for every $n \in \mathbb{N}$, so that $x_n \to 1$ as $n \to \infty$.
- If $a = 0$, then $x_n = 0$ for every $n \in \mathbb{N}$, so that $x_n \to 0$ as $n \to \infty$.
- If $a > 1$, then $a = 1 + k$, where $k > 0$. Then $x_n \to \infty$ as $n \to \infty$, since

$$|a^n| = (1+k)^n \ge 1 + kn > E \quad \text{for every } n > \frac{E-1}{k}.$$

- If $0 < a < 1$, then $a = 1/b$, where $b > 1$. Hence $1/x_n \to \infty$ as $n \to \infty$. It follows that $x_n \to 0$ as $n \to \infty$.
- If $-1 < a < 0$, then $a = -b$, where $0 < b < 1$. We then have $b^n \to 0$ as $n \to \infty$. Also, $-b^n \le x_n \le b^n$ for every $n \in \mathbb{N}$. It follows from the Squeezing principle that $x_n \to 0$ as $n \to \infty$.
- If $a = -1$, then $x_n = (-1)^n$ does not converge as $n \to \infty$.
- If $a < -1$, then $a = 1/b$ where $-1 < b < 0$. Hence $1/x_n \to 0$ as $n \to \infty$. It follows that $x_n \to \infty$ as $n \to \infty$.

Our next task is to study monotonic sequences.

Definition. Let x_n be a real sequence.
(1) We say that x_n is increasing if $x_{n+1} \ge x_n$ for every $n \in \mathbb{N}$.
(2) We say that x_n is decreasing if $x_{n+1} \le x_n$ for every $n \in \mathbb{N}$.
(3) We say that x_n is bounded above if there exists $B \in \mathbb{R}$ such that $x_n \le B$ for every $n \in \mathbb{N}$.
(4) We say that x_n is bounded below if there exists $b \in \mathbb{R}$ such that $x_n \ge b$ for every $n \in \mathbb{N}$.

Remark. Note that a real sequence is bounded if and only if it is bounded above and below.

PROPOSITION 19E. *Suppose that x_n is an increasing real sequence.*
(a) If x_n is bounded above, then x_n converges as $n \to \infty$.
(b) If x_n is not bounded above, then $x_n \to \infty$ as $n \to \infty$.

PROPOSITION 19F. *Suppose that x_n is a decreasing real sequence.*
(a) If x_n is bounded below, then x_n converges as $n \to \infty$.
(b) If x_n is not bounded below, then $x_n \to \infty$ as $n \to \infty$.

Example 19.2.5. The sequence $x_n = 3 - 1/n$ is increasing and bounded above. It is not too difficult that the smallest real number $B \in \mathbb{R}$ such that $x_n \le B$ for every $n \in \mathbb{N}$ is 3. It is easy to show that $x_n \to 3$ as $n \to \infty$.

Example 19.2.6. Consider the sequence

$$x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}.$$

Clearly x_n is an increasing sequence. On the other hand,

$$\begin{aligned} x_n &= 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{(n-1)n} \\ &= 1 + 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 3 - \frac{1}{n} < 3, \end{aligned}$$

so that x_n is bounded above. Unfortunately, it is very hard to find the smallest real number $B \in \mathbb{R}$ such that $x_n \le B$ for every $n \in \mathbb{N}$. While Proposition 19E tells us that the sequence x_n converges, it does not tell us the precise value of the limit. In fact, the limit in this case is the number e.

Example 19.2.7. Consider the sequence $x_n = 1 + a + a^2 + \ldots + a^n$. Then $x_n = n + 1$ if $a = 1$ and

$$x_n = \frac{1 - a^{n+1}}{1 - a} \quad \text{if } a \ne 1.$$

Suppose that $a > 0$. Then x_n is increasing. If $0 < a < 1$, then $x_n < 1/(1 - a)$ for all $n \in \mathbb{N}$, and so x_n converges as $n \to \infty$. If $a \ge 1$, then x_n is not bounded above, so that $x_n \to \infty$ as $n \to \infty$. In fact, if $a \ne 1$, then the convergence or divergence of x_n depends on the convergence and divergence of a^{n+1}, which we have considered before in Example 19.2.4.

19.3. Recurrence Relations

In practice, it may not always be convenient to define a sequence explicitly. Sequences may often be defined by a relation connecting two or more successive terms. Here we shall not make a thorough study of such relations, but confine our discussion to two examples.

EXAMPLE 19.3.1. Suppose that $x_1 = 3$ and

$$x_{n+1} = \frac{4x_n + 2}{x_n + 3}$$

for every $n \in \mathbb{N}$. Note first of all that $0 < x_2 < x_1$. Suppose that $n > 1$ and $0 < x_n < x_{n-1}$. Then clearly $x_{n+1} > 0$. Furthermore,

$$x_{n+1} - x_n = \frac{4x_n + 2}{x_n + 3} - \frac{4x_{n-1} + 2}{x_{n-1} + 3} = \frac{10(x_n - x_{n-1})}{(x_n + 3)(x_{n-1} + 3)} < 0.$$

It follows from the Principle of induction that x_n is a decreasing sequence and bounded below by 0, so that x_n converges as $n \to \infty$. Suppose that $x_n \to x$ as $n \to \infty$. Then

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{4x_n + 2}{x_n + 3} = \frac{4x + 2}{x + 3}.$$

Hence $x = 2$. Note that the other solution $x = -1$ has to be discounted, since $x_n > 0$ for every $n \in \mathbb{N}$.

EXAMPLE 19.3.2. Let $s > 0$. Suppose that $x_1 > 0$ and that for $n > 1$, we have

$$x_n = \frac{1}{2}\left(x_{n-1} + \frac{s}{x_{n-1}}\right).$$

It is not difficult to show that $x_n > 0$ for every $n \in \mathbb{N}$. On the other hand, for $n > 1$, we have

$$x_n^2 = \frac{1}{4}\left(x_{n-1}^2 + \frac{s^2}{x_{n-1}^2} + 2s\right),$$

so that

$$x_n^2 - s = \frac{1}{4}\left(x_{n-1}^2 + \frac{s^2}{x_{n-1}^2} - 2s\right) = \frac{1}{4}\left(x_{n-1} - \frac{s}{x_{n-1}}\right)^2 \geq 0,$$

and so

$$x_{n+1} - x_n = \frac{1}{2}\left(x_n + \frac{s}{x_n}\right) - x_n = \frac{1}{2}\left(\frac{s}{x_n} - x_n\right) = \frac{s - x_n^2}{2x_n} \leq 0.$$

It follows that, with the possible exception that $x_2 \leq x_1$ may not hold, the sequence x_n is decreasing and bounded below, so that x_n converges as $n \to \infty$. Suppose that $x_n \to x$ as $n \to \infty$. Then

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{2}\left(x_{n-1} + \frac{s}{x_{n-1}}\right) = \frac{1}{2}\left(x + \frac{s}{x}\right),$$

so that $x^2 = s$. This gives a proof that s has a square root.

19.4. Further Discussion

In this section, we first give formal proofs of the various results stated in the earlier sections.

PROOF OF PROPOSITION 19A. Suppose that $x_n \to x'$ and $x_n \to x''$ as $n \to \infty$. Then given any $\epsilon > 0$, there exist $N', N'' \in \mathbb{R}$ such that

$$|x_n - x'| < \epsilon \quad \text{whenever } n > N',$$

and

$$|x_n - x''| < \epsilon \quad \text{whenever } n > N''.$$

Let $N = \max\{N', N''\} \in \mathbb{R}$. It follows that whenever $n > N$, we have

$$|x' - x''| \le |x_n - x'| + |x_n - x''| < 2\epsilon.$$

Now $|x' - x''|$ is a non-negative constant less than any $2\epsilon > 0$, so we must have $|x' - x''| = 0$, whence $x' = x''$. ◯

PROOF OF PROPOSITION 19B. Suppose that $x_n \to x$ as $n \to \infty$. Then there exists $N \in \mathbb{N}$ such that $|x_n - x| < 1$ for every $n > N$. Hence

$$|x_n| < |x| + 1 \quad \text{whenever } n > N.$$

Let $M = \max\{|x_1|, \ldots, |x_N|, |x| + 1\}$. Then clearly $|x_n| \le M$ for every $n \in \mathbb{N}$. ◯

PROOF OF PROPOSITION 19C. (a) We shall use the inequality

$$|(x_n + y_n) - (x + y)| \le |x_n - x| + |y_n - y|.$$

Given any $\epsilon > 0$, there exist $N_1, N_2 \in \mathbb{R}$ such that

$$|x_n - x| < \epsilon/2 \quad \text{whenever } n > N_1,$$

and

$$|y_n - y| < \epsilon/2 \quad \text{whenever } n > N_2.$$

Let $N = \max\{N_1, N_2\} \in \mathbb{R}$. It follows that whenever $n > N$, we have

$$|(x_n + y_n) - (x + y)| \le |x_n - x| + |y_n - y| < \epsilon.$$

(b) We shall use the inequality

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\ &= |x_n(y_n - y) + (x_n - x)y| \\ &\le |x_n||y_n - y| + |y||x_n - x|. \end{aligned}$$

Since $x_n \to x$ as $n \to \infty$, there exists $N_1 \in \mathbb{R}$ such that

$$|x_n - x| < 1 \quad \text{whenever } n > N_1,$$

so that

$$|x_n| < |x| + 1 \quad \text{whenever } n > N_1.$$

On the other hand, given any $\epsilon > 0$, there exist $N_2, N_3 \in \mathbb{R}$ such that

$$|x_n - x| < \frac{\epsilon}{2(|y|+1)} \quad \text{whenever } n > N_2,$$

and

$$|y_n - y| < \frac{\epsilon}{2(|x|+1)} \quad \text{whenever } n > N_3.$$

Let $N = \max\{N_1, N_2, N_3\} \in \mathbb{R}$. It follows that whenever $n > N$, we have

$$|x_n y_n - xy| \le |x_n||y_n - y| + |y||x_n - x| < \epsilon.$$

(c) We shall first show that $1/y_n \to 1/y$ as $n \to \infty$. To do this, we shall use the identity

$$\left| \frac{1}{y_n} - \frac{1}{y} \right| = \frac{|y_n - y|}{|y_n||y|}.$$

Since $y \ne 0$ and $y_n \to y$ as $n \to \infty$, there exists $N_1 \in \mathbb{R}$ such that

$$|y_n - y| < |y|/2 \quad \text{whenever } n > N_1,$$

so that

$$|y_n| > |y|/2 \quad \text{whenever } n > N_1.$$

On the other hand, given any $\epsilon > 0$, there exists $N_2 \in \mathbb{R}$ such that

$$|y_n - y| < y^2\epsilon/2 \quad \text{whenever } n > N_2.$$

Let $N = \max\{N_1, N_2\} \in \mathbb{R}$. It follows that whenever $n > N$, we have

$$\left| \frac{1}{y_n} - \frac{1}{y} \right| = \frac{|y_n - y|}{|y_n||y|} \le \frac{2|y_n - y|}{|y|^2} < \epsilon.$$

We now apply part (b) to x_n and $1/y_n$ to get the desired result. ○

Proof of Proposition 19D. By Proposition 19C, $y_n - x_n \to 0$ as $n \to \infty$. It follows that given any $\epsilon > 0$, there exist $N', N'' \in \mathbb{R}$ such that

$$|y_n - x_n| < \epsilon/2 \quad \text{whenever } n > N',$$

and

$$|x_n - x| < \epsilon/2 \quad \text{whenever } n > N''.$$

Let $N = \max\{N', N''\} \in \mathbb{R}$. It follows that whenever $n > N$, we have

$$|a_n - x| \le |a_n - x_n| + |x_n - x| \le |y_n - x_n| + |x_n - x| < \epsilon.$$

Hence $a_n \to x$ as $n \to \infty$. ○

PROOF OF PROPOSITION 19E. (a) Suppose that the sequence x_n is bounded above. Then the set

$$S = \{x_n : n \in \mathbb{N}\}$$

is a non-empty set of real numbers which is bounded above. Let $x = \sup S$. We shall show that $x_n \to x$ as $n \to \infty$. Given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $x_N > x - \epsilon$. Since the sequence x_n is increasing and bounded above by x, it follows that whenever $n > N$, we have $x \geq x_n \geq x_N > x - \epsilon$, so that $|x_n - x| < \epsilon$.

(b) Suppose that the sequence x_n is not bounded above. Then for every $E > 0$, there exists $N \in \mathbb{N}$ such that $x_N > E$. Since the sequence x_n is increasing, it follows that $|x_n| = x_n \geq x_N > E$ for every $n > N$. Hence $x_n \to \infty$ as $n \to \infty$. ◯

We conclude this chapter by discussing subsequences. Heuristically, a subsequence is obtained from a sequence by possibly omitting some of the terms, and keeping the remainder in the original order. We can make this more formal in the following way.

DEFINITION. Suppose that

$$x_1, x_2, x_3, \ldots, x_n, \ldots$$

is a real sequence. Suppose further that $n_1 < n_2 < n_3 < \ldots < n_p < \ldots$ is an infinite sequence of natural numbers. Then the sequence

$$x_{n_1}, x_{n_2}, x_{n_3}, \ldots, x_{n_p}, \ldots$$

is called a subsequence of the original sequence.

EXAMPLE 19.4.1. The sequence $2, 4, 6, 8, \ldots$ of even natural numbers is a subsequence of the sequence $1, 2, 3, 4, \ldots$ of natural numbers.

EXAMPLE 19.4.2. The sequence $2, 3, 5, 7, \ldots$ of primes is not a subsequence of the sequence $1, 3, 5, 7, \ldots$ of odd natural numbers.

EXAMPLE 19.4.3. The sequence $1, 2, 3, 4, \ldots$ of natural numbers is a subsequence of the sequence $\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots$.

We shall establish the following important result in analysis.

PROPOSITION 19G. *Every bounded sequence of real numbers has a convergent subsequence.*

PROOF. We say that $n \in \mathbb{N}$ is a "peak" point if $x_n > x_m$ for every $m > n$. There are two possibilities:

(i) Suppose that there are infinitely many peak points $n_1 < n_2 < n_3 < \ldots < n_p < \ldots$. Then

$$x_{n_1} > x_{n_2} > x_{n_3} > \ldots > x_{n_p} > \ldots$$

is a decreasing subsequence, clearly bounded below, and is therefore convergent by Proposition 19F.

(ii) Suppose that there are no or only finitely many peak points. Let $n_1 = 1$ if there are no peak points, and let $n_1 = N + 1$ if N represents the largest peak point. Then n_1 is not a peak point, and so there exists $n_2 > n_1$ such that $x_{n_1} \leq x_{n_2}$. On the other hand, n_2 is not a peak point, and so there exists $n_3 > n_2$ such that $x_{n_2} \leq x_{n_3}$. Continuing inductively, we conclude that there exists an infinite sequence $n_1 < n_2 < n_3 < \ldots < n_p < \ldots$ of natural numbers such that

$$x_{n_1} \leq x_{n_2} \leq x_{n_3} \leq \ldots \leq x_{n_p} \leq \ldots$$

is an increasing subsequence, clearly bounded above, and is therefore convergent by Proposition 19E. ◯

Problems for Chapter 19

1. Use the ϵ-N definition to prove each of the following convergence as $n \to \infty$:
   a) $x_n = \dfrac{3n+7}{2n+9} \to \dfrac{3}{2}$ b) $x_n = \dfrac{3}{n^2} \to 0$

2. Use the arithmetic of limits to find the limit of each of the following sequences:
   a) $x_n = \dfrac{n^2+1}{n^2+5}$ b) $x_n = \dfrac{3n^2+4n+5}{2n^2-3n+7}$

3. Use the Squeezing principle to find the limit of each of the following sequences:
   a) $x_n = \dfrac{1}{n}\sin\dfrac{n\pi}{3}\cos\dfrac{n\pi}{4}$ b) $x_n = \begin{cases} 1/n & \text{if } n \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$
   c) $x_n = \begin{cases} 1/n & \text{if } n \text{ is odd} \\ -1/n^2 & \text{if } n \text{ is even} \end{cases}$

4. Find the limit of each of the following sequences, and try to justify your assertions:
   a) $x_n = \dfrac{n}{2^n}$ b) $x_n = \dfrac{1+2+\ldots+n}{n^2}$
   c) $x_n = \dfrac{(-1)^n}{4n+3} + \dfrac{2n+1}{3n+2}$ d) $x_n = \dfrac{1}{2^n} + \dfrac{3n+4}{2n+9}$
   e) $x_n = \dfrac{2}{n} + \dfrac{(-1)^n}{n^2}$ f) $x_n = \dfrac{2n+3}{3n^2+4}\cos\dfrac{n\pi}{6}$
   g) $x_n = \begin{cases} n^2 & \text{if } n \le 10 \\ 1/n & \text{if } n > 10 \end{cases}$

5. For what values of $a, b \in \mathbb{R}$ does the sequence $x_n = a + b(-1)^n$ converge?

6. Find a real sequence x_n that satisfies the following conditions simultaneously:
   a) $0 < x_n < 1$ for every $n \in \mathbb{N}$;
   b) $x_n \neq 1/2$ for every $n \in \mathbb{N}$; and
   c) $x_n \to 1/2$ as $n \to \infty$.

7. Suppose that x is a real number. Discuss the convergence of the sequence $x_n = \dfrac{x+x^n}{1+x^n}$, taking care to distinguish the four cases $|x| > 1$, $|x| < 1$, $x = 1$ and $x = -1$.

Harder Problems for Chapter 19

8. A sequence x_n is defined inductively by $x_1 = 1$ and $x_{n+1} = \sqrt{x_n + 6}$ for every $n \in \mathbb{N}$.
   a) Prove by induction that x_n is increasing, and $x_n < 3$ for every $n \in \mathbb{N}$.
   b) Deduce that x_n converges as $n \to \infty$ and find its limit.

9. Suppose that $x_1 < x_2$ and $x_{n+2} = \frac{1}{2}(x_{n+1} + x_n)$ for every $n \in \mathbb{N}$. Show that
   a) $x_{n+2} > x_n$ for every odd $n \in \mathbb{N}$;
   b) $x_{n+2} < x_n$ for every even $n \in \mathbb{N}$; and
   c) $x_n \to \frac{1}{3}(x_1 + 2x_2)$ as $n \to \infty$.

10. Suppose that $a_n \to L$ as $n \to \infty$, and that $s_n = \dfrac{1}{n}(a_1 + \ldots + a_n)$ for every $n \in \mathbb{N}$. Show that $s_n \to L$ as $n \to \infty$.
    [Hint: Consider first the case $L = 0$.]

11. Show that the sequence $x_n = \left(1 + \dfrac{1}{n}\right)^n$ is increasing and bounded above.

    [Remark: Hence it converges. The limit is e.]

12. For each of the following sequences x_n, find monotonic subsequences:

    a) $x_n = an + b$

    b) $x_n = \begin{cases} 1 & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}$

    c) $x_n = \cos \dfrac{3n\pi}{4}$

    d) $x_n = \begin{cases} 1/n & \text{if } n \text{ prime} \\ 0 & \text{otherwise} \end{cases}$