FIRST YEAR CALCULUS

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Chapter 20

SERIES

20.1. Introduction

In this chapter, we are concerned with expressions of the type

$$\sum_{n=1}^{\infty} x_n = x_1 + x_2 + x_3 + \ldots, \tag{1}$$

where $x_n \in \mathbb{R}$ for every $n \in \mathbb{N}$.

Before we proceed in any formal way, let us examine three examples.

Example 20.1.1. Consider the expression

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots.$$

We shall try to interpret this by looking at a practical situation. Consider a square of area 1. Let us first of all shade half of it, then half of what remains, then half of what remains, and so on. Note that we are shading parts of area 1/2, 1/4, 1/8, and so on. Since at every stage, we are shading half of what remains, the total area of the shaded part will get closer to 1 the longer we keep at it. More precisely, after N steps, the shaded part will have area

$$s_N = \sum_{n=1}^{N} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^N} = 1 - \frac{1}{2^N}.$$

Note that $s_N \to 1$ as $N \to \infty$. It is therefore reasonable to say that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Example 20.1.2. Consider the expression

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots.$$

For every $N \in \mathbb{N}$, write

$$t_N = \sum_{n=1}^{N} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{N}.$$

Then it is easy to see that t_N is an increasing sequence. Is t_N bounded above? Let us examine some special values of N. Suppose that $N = 2^m$ for some $m \in \mathbb{N}$. Then

$$\begin{aligned} t_{2^m} &= \sum_{n=1}^{2^m} \frac{1}{n} \\ &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \ldots + \frac{1}{8}\right) + \left(\frac{1}{9} + \ldots + \frac{1}{16}\right) + \ldots + \left(\frac{1}{2^{m-1}+1} + \ldots + \frac{1}{2^m}\right) \\ &> 1 + \frac{1}{2} + (4-2)\frac{1}{4} + (8-4)\frac{1}{8} + (16-8)\frac{1}{16} + \ldots + (2^m - 2^{m-1})\frac{1}{2^m} = 1 + \frac{m}{2}. \end{aligned}$$

It follows that t_N can be made as large as we please by choosing N large enough, so that t_N is not bounded above. Since t_N is increasing, it follows that $t_N \to \infty$ as $N \to \infty$. This means that the expression

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is infinite.

Example 20.1.3. Consider the expression

$$\sum_{n=1}^{\infty} (-1)^{n-1} = 1 - 1 + 1 - 1 + \ldots.$$

For every $N \in \mathbb{N}$, write

$$s_N = \sum_{n=1}^{N} (-1)^{n-1}.$$

Then $s_1 = s_3 = s_5 = \ldots = 1$ and $s_2 = s_4 = s_6 = \ldots = 0$. It follows that the sequence s_N does not converge, so that we cannot attach any value, finite or infinite, to the expression

$$\sum_{n=1}^{\infty} (-1)^{n-1} = 1 - 1 + 1 - 1 + \ldots.$$

We are now in a reasonable position to formulate a definition.

DEFINITION. For every $N \in \mathbb{N}$, the expression

$$s_N = \sum_{n=1}^{N} x_n$$

is called the N-th partial sum of the series (1). If s_N converges to a finite limit s as $N \to \infty$, then we say that the series (1) is convergent with sum s, and write

$$\sum_{n=1}^{\infty} x_n = s.$$

If s_N diverges as $N \to \infty$, then we say that the series (1) is divergent.

REMARK. Since the convergence or divergence of a series is determined by the convergence or divergence of the sequence of partial sums, we can use techniques for sequences to study the sequence of partial sums. Indeed, we have used this approach in our three examples so far.

EXAMPLE 20.1.4. The series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

is convergent with sum 1.

EXAMPLE 20.1.5. The series

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{and} \qquad \sum_{n=1}^{\infty} (-1)^{n-1}$$

both diverge.

REMARK. It is not necessary to start the series with $n = 1$. In fact, in many instances, it is convenient to study series of the form

$$\sum_{n=0}^{\infty} x_n = x_0 + x_1 + x_2 + x_3 + \ldots.$$

The convention is that if we consider the series

$$\sum_{n=k}^{\infty} x_n,$$

then for every $N \in \mathbb{N}$ satisfying $N \geq k$, we write

$$s_N = \sum_{n=k}^{N} x_n.$$

For the remainder of this section, we shall discuss a few very basic results concerning convergence of series. The proofs are very simple and are included here. However, they depend on knowledge on sequences. Before going any further, the reader is advised to study Chapter 19 again in detail.

PROPOSITION 20A. *The convergence or divergence of a series is unaffected if a finite number of terms are inserted, deleted or altered.*

Proof. Note that if N_0 is large enough, then all insertions, deletions or alterations will occur before the N_0-th term. It follows that for every $N \in \mathbb{N}$ such that $N > N_0$, the partial sum s_N has been altered by a fixed finite amount, and this does not affect the convergence or divergence of the sequence s_N. ○

PROPOSITION 20B. *Suppose that*

$$\sum_{n=1}^{\infty} x_n = s \qquad \textit{and} \qquad \sum_{n=1}^{\infty} y_n = t. \tag{2}$$

Then for every $a, b \in \mathbb{R}$, we have

$$\sum_{n=1}^{\infty} (ax_n + by_n) = as + bt. \tag{3}$$

Proof. If s_N and t_N represent the sequences of partial sums of the two series in (2) respectively, then $as_N + bt_N$ represents the sequence of partial sums of the series in (3). ○

PROPOSITION 20C. *Suppose that the series* (1) *is convergent. Then $x_n \to 0$ as $n \to \infty$.*

Proof. Note that $x_n = s_n - s_{n-1} \to s - s = 0$ as $n \to \infty$. ○

Remark. Suppose that $x_n = 1/n$. Note that $x_n \to 0$ as $n \to \infty$. Note also that the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent. Compare this to Proposition 20C.

In fact, Proposition 20C is more useful if stated in the following equivalent form.

PROPOSITION 20D. *Suppose that the sequence x_n does not converge to 0 as $n \to \infty$. Then the series* (1) *is divergent.*

Example 20.1.6. The series

$$\sum_{n=1}^{\infty} \frac{2n+3}{3n+4}$$

is divergent, since the sequence

$$\frac{2n+3}{3n+4} \to \frac{2}{3} \quad \text{as } n \to \infty.$$

20.2. Some Well Known Series

In this section, we shall study two well known series which underpin much of the discussion on convergence and divergence of many other series.

PROPOSITION 20E. *Suppose that $a \in \mathbb{R}$. Then the geometric series*

$$\sum_{n=1}^{\infty} a^{n-1} = 1 + a + a^2 + a^3 + \ldots$$

converges if and only if $|a| < 1$.

Proof. Consider the sequence of partial sums

$$s_N = \sum_{n=1}^{N} a^{n-1} = 1 + a + a^2 + a^3 + \ldots + a^{N-1}.$$

Then $s_N = N$ if $a = 1$ and

$$s_N = \frac{1 - a^N}{1 - a} \quad \text{if } a \neq 1.$$

If $|a| < 1$, then $a^N \to 0$ as $N \to \infty$, so that the series is convergent with sum $(1-a)^{-1}$. If $|a| \geq 1$, then a^{n-1} does not converge to 0 as $n \to \infty$, so that the series in divergent in view of Proposition 20D. ○

In Section 20.7, we shall establish the following important result concerning harmonic series.

PROPOSITION 20F. *Suppose that $p \in \mathbb{R}$. Then the series*

$$\sum_{n=1}^{\infty} n^{-p}$$

is convergent if $p > 1$ and divergent if $p \leq 1$.

Example 20.2.1. The series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent. It can be shown that its sum is equal to $\pi^2/6$.

Example 20.2.2. The series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

is convergent. Its sum is usually denoted by $\zeta(3)$. It was a major achievement in number theory when Roger Apéry showed that $\zeta(3)$ is irrational.

20.3. Series of Non-Negative Terms

The ideas in the following two results are used in the proof of Proposition 20F. The simple proofs are included here.

PROPOSITION 20G. *Suppose that $x_n \geq 0$ for every $n \in \mathbb{N}$. Then either the series* (1) *converges, or its sequence of partial sums diverges to infinity.*

Proof. Note that the sequence of partial sums form an increasing sequence. The result now follows from Proposition 19E. ○

PROPOSITION 20H. (COMPARISON TEST) *Suppose that for every $n \in \mathbb{N}$, we have $x_n \geq 0$, $y_n \geq 0$ and $x_n \leq Cy_n$, where C is a fixed positive constant. If the series*

$$\sum_{n=1}^{\infty} y_n \tag{4}$$

is convergent, then the series

$$\sum_{n=1}^{\infty} x_n \tag{5}$$

is convergent. On the other hand, if the series (5) *is divergent, then the series* (4) *is divergent.*

PROOF. Note that the second assertion follows from the first. To prove the first assertion, let

$$s_N = \sum_{n=1}^{N} x_n \qquad \text{and} \qquad t_N = \sum_{n=1}^{N} y_n$$

denote the sequences of partial sums of the series. Then clearly s_N and t_N are increasing sequences. If the series (4) is convergent, then t_N converges and so is bounded above. Since $s_N \leq Ct_N$ for every $N \in \mathbb{N}$, it follows that s_N is bounded above. It follows from Proposition 19E that the series (5) is convergent. ◯

EXAMPLE 20.3.1. Consider the series

$$\sum_{n=1}^{\infty} \frac{2^{-n}}{n^{3/2}}.$$

Since $2^{-n} \leq 1$ for every $n \in \mathbb{N}$, it follows that

$$\frac{2^{-n}}{n^{3/2}} \leq \frac{1}{n^{3/2}}$$

for every $n \in \mathbb{N}$. On the other hand, it follows from Proposition 20F that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

is convergent. It therefore follows from the Comparison test that the original series is convergent.

20.4. Conditional Convergence

EXAMPLE 20.4.1. Recall that the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

diverges. Let us now consider the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots. \tag{6}$$

Denote the partial sum by

$$s_N = \sum_{n=1}^{N} (-1)^{n-1} \frac{1}{n}.$$

Then it is not too difficult to see that for every $m \in \mathbb{N}$, we have

$$s_1 \geq s_3 \geq s_5 \geq \ldots \geq s_{2m-1} \geq s_{2m} \geq \ldots \geq s_6 \geq s_4 \geq s_2.$$

It follows that the sequence $s_1, s_3, s_5, \ldots$ is decreasing and bounded below by s_2, while the sequence $s_2, s_4, s_6, \ldots$ is increasing and bounded above by s_1. So both sequences converge. Note also that

$$s_{2m-1} - s_{2m} = \frac{1}{2m} \to 0$$

as $m \to \infty$, so that the two sequences converge to the same limit. This means that the sequence s_N converges as $N \to \infty$, so that the series (6) is convergent.

A similar argument will establish the following result. The proof will be given in Section 20.7.

PROPOSITION 20J. (ALTERNATING SERIES TEST) *Suppose that*
(a) $a_n > 0$ *for every* $n \in \mathbb{N}$*;*
(b) a_n *is a decreasing sequence; and*
(c) $a_n \to 0$ *as* $n \to \infty$*.*
Then the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

is convergent.

REMARK. It is quite clear that the convergence of the series (6) is due entirely to the fact that there is sufficient cancellation between positive and negative terms.

EXAMPLE 20.4.2. The logarithmic series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

is convergent (with sum $\log 2$) if $x = 1$ and divergent if $x = -1$.

20.5. Absolute Convergence

EXAMPLE 20.5.1. We have just shown that the series (6) is convergent. Let s be its sum. In other words, let

$$s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$$

Let us now rearrange the terms and consider the series

$$\begin{aligned} 1 &- \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \ldots \\ &= \left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + \ldots \\ &= \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{10} - \frac{1}{12}\right) + \ldots = \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}\right) = \frac{s}{2}. \end{aligned}$$

Note that no term has been omitted or inserted in the rearrangement. Note also that $s \neq 0$. But yet we end up with a different sum. The only possible explanation is that the convergence of the original and the rearranged series depend on cancallation between positive and negative terms. The difference therefore has to arise from the nature of such cancellation.

Suppose now that the convergence of a series does not depend on the cancellation between positive and negative terms. Then it is reasonable to ask whether any rearrangement of the terms may still alter the sum of the series.

The first step towards an answer to this question is summarized below. See Section 20.7 for a proof.

PROPOSITION 20K. it Suppose that the series

$$\sum_{n=1}^{\infty} |x_n| \tag{7}$$

converges. Then the series

$$\sum_{n=1}^{\infty} x_n \tag{8}$$

converges. Furthermore, we have

$$\left|\sum_{n=1}^{\infty} x_n\right| \le \sum_{n=1}^{\infty} |x_n|.$$

EXAMPLE 20.5.2. Let $C > 0$ be a constant. Suppose that $|a(n)| \le C$ for every $n \in \mathbb{N}$. Then

$$\frac{|a(n)|}{n^2} \le C\frac{1}{n^2}$$

for every $n \in \mathbb{N}$. Since the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent, it follows from the Comparison test that the series

$$\sum_{n=1}^{\infty} \left|\frac{a(n)}{n^2}\right| = \sum_{n=1}^{\infty} \frac{|a(n)|}{n^2}$$

is convergent. It now follows from Proposition 20K that the series

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^2}$$

is convergent.

DEFINITION. We say that the series (8) is absolutely convergent if the series (7) is convergent.

REMARK. Proposition 20K essentially states that every absolutely convergent series is convergent.

The Comparison test can now be stated in the following stronger form, in view of Proposition 20K.

PROPOSITION 20L. (COMPARISON TEST) *Suppose that for every $n \in \mathbb{N}$, we have $y_n \geq 0$ and $|x_n| \leq C y_n$, where C is a fixed positive constant. If the series*

$$\sum_{n=1}^{\infty} y_n$$

is convergent, then the series

$$\sum_{n=1}^{\infty} x_n$$

is absolutely convergent.

The Comparison test is one of the most important results in the study of convergence of series. In particular, the following two important tests for convergence are established by comparing the series in question with artificially constructed convergent geometric series.

PROPOSITION 20M. (RATIO TEST) *Suppose that the sequence x_n satisfies*

$$\left| \frac{x_{n+1}}{x_n} \right| \to l \quad \text{as } n \to \infty.$$

Then the series

$$\sum_{n=1}^{\infty} x_n$$

is absolutely convergent if $l < 1$ and divergent if $l > 1$.

PROPOSITION 20N. (ROOT TEST) *Suppose that the sequence x_n satisfies*

$$|x_n|^{1/n} \to l \quad \text{as } n \to \infty.$$

Then the series

$$\sum_{n=1}^{\infty} x_n$$

is absolutely convergent if $l < 1$ and divergent if $l > 1$.

REMARK. No firm conclusion can be drawn if $l = 1$. In the case of the Ratio test, consider the two series

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

It is easy to show that $l = 1$ in both cases. Note from Proposition 20F that the first series is divergent while the second series is convergent.

EXAMPLE 20.5.3. Consider the series

$$\sum_{n=1}^{\infty} \frac{n!(3n)!}{(4n)!} 9^n.$$

Here

$$x_n = \frac{n!(3n)!}{(4n)!} 9^n,$$

so that

$$\left|\frac{x_{n+1}}{x_n}\right| = \frac{(n+1)!(3n+3)!(4n)!9^{n+1}}{n!(3n)!(4n+4)!9^n} = \frac{9(n+1)(3n+3)(3n+2)(3n+1)}{(4n+4)(4n+3)(4n+2)(4n+1)} \to \frac{243}{256}$$

as $n \to \infty$. Hence the series is absolutely convergent.

Example 20.5.4. Consider the series

$$\sum_{n=1}^{\infty} n^p a^n$$

where $p \in \mathbb{Z}$ and $a \in \mathbb{R}$ are fixed. If $a = 0$, then clearly the series is convergent, so we assume that $a \neq 0$. Here $x_n = n^p a^n$, so that

$$\left|\frac{x_{n+1}}{x_n}\right| = \left|\frac{(n+1)^p a^{n+1}}{n^p a^n}\right| = \left(\frac{n+1}{n}\right)^p |a| \to |a|$$

as $n \to \infty$. Hence the series is absolutely convergent if $|a| < 1$ and divergent if $|a| > 1$. If $a = 1$, then $x_n = n^p$, and we can appeal to Proposition 20F. If $a = -1$, then $x_n = (-1)^n n^p$. We have two cases. If $p \geq 0$, then $|x_n| \not\to 0$ as $n \to \infty$, and we can appeal to Proposition 20D to conclude that the series is divergent. If $p < 0$, then the sequence n^p decreases to the limit 0 as $n \to \infty$, and we can appeal to the Alternating series test to conclude that the series is convergent.

We conclude this section by answering the question first raised at the beginning of this section. See Section 20.7 for a proof of the result below.

PROPOSITION 20P. *Any rearrangement of an absolutely convergent series does not alter its sum.*

20.6. Relationship with Integrals

Quite often, the question of the convergence or divergence of a series can be translated to a question of the convergence or divergence of some improper integrals. Here we mention one of the simplest cases. The proof can be found in Section 20.7.

PROPOSITION 20Q. *Suppose that*

(a) $f(x) > 0$ for every $x \in \mathbb{R}$; and

(b) $f(x)$ is a decreasing function for $x \geq 1$, so that for every $x_1, x_2 \in \mathbb{R}$ satisfying $1 \leq x_1 < x_2$, we have $f(x_1) \geq f(x_2)$.

Then the sequence

$$\sigma_N = \sum_{n=1}^{N} f(n) - \int_1^N f(x)\,\mathrm{d}x$$

is a decreasing sequence and converges to a limit $\sigma \in \mathbb{R}$ as $N \to \infty$, where $0 \leq \sigma \leq f(1)$. Furthermore,

$$\sum_{n=1}^{\infty} f(n) \qquad \textit{and} \qquad \int_1^{\infty} f(x)\,\mathrm{d}x$$

are either both convergent or both divergent.

EXAMPLE 20.6.1. Let $p \in \mathbb{R}$ and $k \in \mathbb{N}$. If k is sufficiently large, then the series

$$\sum_{n=k}^{\infty} \frac{1}{n^p} \qquad \text{and} \qquad \sum_{n=k}^{\infty} \frac{1}{n(\log n)^p} \qquad \text{and} \qquad \sum_{n=k}^{\infty} \frac{1}{n \log n (\log\log n)^p}$$

are all convergent if $p > 1$ and all divergent if $p \leq 1$.

EXAMPLE 20.6.2. It follows from Proposition 20Q that

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{N} - \log N = \sum_{n=1}^{N} \frac{1}{n} - \int_1^N \frac{\mathrm{d}x}{x} \to \gamma$$

as $n \to \infty$, where $0 \leq \gamma \leq 1$. The number γ is called Euler's constant. It is not known whether γ is rational or irrational.

20.7. Further Discussion

In this section, we shall give the proofs of a number of results discussed earlier.

PROOF OF PROPOSITION 20F. Consider the sequence of partial sums

$$s_N = \sum_{n=1}^{N} n^{-p}.$$

Clearly s_N is an increasing sequence. We shall use Proposition 19E.

(a) For $p = 1$, we have already shown that the sequence

$$t_N = \sum_{n=1}^{N} n^{-1} \to \infty$$

as $N \to \infty$, so that the series diverges.

(b) Suppose now that $p < 1$. Note that for every $N \in \mathbb{N}$, we have $s_N \geq t_N$. It follows that

$$0 < \frac{1}{s_N} \leq \frac{1}{t_N}.$$

Note now that $1/t_N \to 0$ as $N \to \infty$. It follows from the Squeezing principle that $1/s_N \to 0$ as $N \to \infty$, so that $s_N \to \infty$ as $N \to \infty$, whence the series diverges.

(c) Suppose now that $p > 1$. It is enough to show that s_N is bounded above. Let $t \in \mathbb{N}$ satisfy $N \leq 2^t - 1$. Then

$$\begin{aligned} s_N \leq s_{2^t-1} &= 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{(2^t-1)^p} \\ &= 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \ldots + \frac{1}{7^p}\right) + \left(\frac{1}{8^p} + \ldots + \frac{1}{15^p}\right) + \ldots + \left(\frac{1}{(2^{t-1})^p} + \ldots + \frac{1}{(2^t-1)^p}\right) \\ &< 1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \ldots + \frac{2^{t-1}}{(2^{t-1})^p} = 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \left(\frac{1}{2^{p-1}}\right)^3 + \ldots + \left(\frac{1}{2^{p-1}}\right)^{t-1} < B, \end{aligned}$$

where

$$B = 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \left(\frac{1}{2^{p-1}}\right)^3 + \ldots = \sum_{n=1}^{\infty} \left(\frac{1}{2^{p-1}}\right)^{n-1}$$

is the sum of a convergent geometric series. ◯

PROOF OF PROPOSITION 20J. Consider the sequence of partial sums

$$s_N = \sum_{n=1}^{N} (-1)^{n-1} a_n.$$

In view of conditions (a) and (b), it is not too difficult to see that for every $m \in \mathbb{N}$, we have

$$s_1 \geq s_3 \geq s_5 \geq \ldots \geq s_{2m-1} \geq s_{2m} \geq \ldots \geq s_6 \geq s_4 \geq s_2.$$

It follows that the sequence $s_1, s_3, s_5, \ldots$ is decreasing and bounded below by s_2, while the sequence $s_2, s_4, s_6, \ldots$ is increasing and bounded above by s_1. So both sequences converge. Note also that in view of condition (c), we have

$$s_{2m-1} - s_{2m} = a_{2m} \to 0$$

as $m \to \infty$, so that the two sequences converge to the same limit. Hence the sequence s_N converges as $N \to \infty$. ◯

PROOF OF PROPOSITION 20K. For every $n \in \mathbb{N}$, we clearly have $x_n = x_n^+ - x_n^-$, where

$$x_n^+ = \begin{cases} x_n & \text{if } x_n \geq 0, \\ 0 & \text{if } x_n < 0, \end{cases}$$

and

$$x_n^- = \begin{cases} 0 & \text{if } x_n \geq 0, \\ -x_n & \text{if } x_n < 0. \end{cases}$$

Furthermore, $0 \leq x_n^+ \leq |x_n|$ and $0 \leq x_n^- \leq |x_n|$ for every $n \in \mathbb{N}$. It follows from the Comparison test that

$$\sum_{n=1}^{\infty} x_n^+ \qquad \text{and} \qquad \sum_{n=1}^{\infty} x_n^-$$

are both convergent. It now follows from Proposition 20B that

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} (x_n^+ - x_n^-)$$

is convergent. To prove the second assertion, write, for every $N \in \mathbb{N}$,

$$T_N = \sum_{n=1}^{N} |x_n| - \left|\sum_{n=1}^{N} x_n\right|.$$

Then it can be shown that T_N is a non-negative convergent sequence. Hence

$$0 \leq \lim_{N\to\infty} T_N = \sum_{n=1}^{\infty} |x_n| - \left|\sum_{n=1}^{\infty} x_n\right|.$$

The second assertion follows. ◯

Proof of Proposition 20M. Suppose first of all that $l < 1$. Let $L = \frac{1}{2}(1+l)$. Clearly $l < L < 1$. Since

$$\left|\frac{x_{n+1}}{x_n}\right| \to l \quad \text{as } n \to \infty,$$

there exists an integer N such that

$$\left|\frac{x_{n+1}}{x_n}\right| < L \quad \text{whenever } n \ge N.$$

It follows that

$$|x_n| < \frac{|x_N|}{L^N} L^n \quad \text{whenever } n > N.$$

On the other hand, the geometric series

$$\sum_{n=1}^{\infty} L^n$$

is convergent. It follows from Comparison test, using Proposition 20A if necessary, that the series

$$\sum_{n=1}^{\infty} |x_n|$$

is convergent. Suppose next that $l > 1$. Then clearly $|x_n| \not\to 0$ as $n \to \infty$. The result follows from Proposition 20D. ○

Proof of Proposition 20N. Suppose first of all that $l < 1$. Let $L = \frac{1}{2}(1+l)$. Clearly $l < L < 1$. Since

$$|x_n|^{1/n} \to l \quad \text{as } n \to \infty,$$

there exists an integer N such that

$$|x_n|^{1/n} < L \quad \text{whenever } n > N.$$

It follows that

$$|x_n| < L^n \quad \text{whenever } n > N.$$

On the other hand, the geometric series

$$\sum_{n=1}^{\infty} L^n$$

is convergent. It follows from Comparison test, using Proposition 20A if necessary, that the series

$$\sum_{n=1}^{\infty} |x_n|$$

is convergent. Suppose next that $l > 1$. Then clearly $|x_n| \not\to 0$ as $n \to \infty$. The result follows from Proposition 20D. ○

Proof of Proposition 20P. Suppose that the series

$$\sum_{n=1}^{\infty} x_n$$

converges absolutely, and that the sequence y_n is a rearrangement of the sequence x_n. We now define $x_n^+, x_n^-, y_n^+, y_n^-$ as in the same way as in the proof of Proposition 20K. Then y_n^+ is a rearrangement of x_n^+ and y_n^- is a rearrangement of x_n^-. Clearly the series

$$\sum_{n=1}^{\infty} x_n^+$$

is convergent. Also, the sequence

$$\sum_{n=1}^{N} y_n^+$$

is increasing and bounded above by

$$\sum_{n=1}^{\infty} x_n^+,$$

so that

$$\sum_{n=1}^{\infty} y_n^+ \le \sum_{n=1}^{\infty} x_n^+.$$

Arguing in the opposite way, we must have

$$\sum_{n=1}^{\infty} x_n^+ \le \sum_{n=1}^{\infty} y_n^+.$$

Hence we must have

$$\sum_{n=1}^{\infty} y_n^+ = \sum_{n=1}^{\infty} x_n^+.$$

Similarly,

$$\sum_{n=1}^{\infty} y_n^- = \sum_{n=1}^{\infty} x_n^-.$$

It now follows that

$$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} y_n^+ - \sum_{n=1}^{\infty} y_n^- = \sum_{n=1}^{\infty} x_n^+ - \sum_{n=1}^{\infty} x_n^- = \sum_{n=1}^{\infty} x_n,$$

and the proof is complete. ◯

Proof of Proposition 20Q. Note first of all that

$$\sigma_{N+1} - \sigma_N = f(N+1) - \int_N^{N+1} f(x)\,\mathrm{d}x = \int_N^{N+1} (f(N+1) - f(x))\,\mathrm{d}x \le 0,$$

since $f(N+1) \le f(x)$ whenever $N \le x \le N+1$. Next, note that

$$\begin{aligned}\sigma_N &= f(1) + \sum_{n=2}^{N} f(n) - \int_1^N f(x)\,\mathrm{d}x = f(1) + \sum_{n=2}^{N} \left(f(n) - \int_{n-1}^{n} f(x)\,\mathrm{d}x \right) \\ &= f(1) + \sum_{n=2}^{N} \int_{n-1}^{n} (f(n) - f(x))\,\mathrm{d}x \le f(1)\end{aligned}$$

and

$$\begin{aligned}\sigma_N &= f(N) + \sum_{n=1}^{N-1} f(n) - \int_1^N f(x)\,\mathrm{d}x = f(N) + \sum_{n=1}^{N-1} \left(f(n) - \int_{n}^{n+1} f(x)\,\mathrm{d}x \right) \\ &= f(N) + \sum_{n=1}^{N-1} \int_{n}^{n+1} (f(n) - f(x))\,\mathrm{d}x \ge f(N) \ge 0.\end{aligned}$$

Hence σ_N is a decreasing sequence bounded below. It follows from Proposition 19F that σ_N converges to some number $\sigma \in \mathbb{R}$ as $N \to \infty$. Since $0 \le \sigma_N \le f(1)$ for every $N \in \mathbb{N}$, we must have $0 \le \sigma \le f(1)$. Finally, if we write

$$s_N = \sum_{n=1}^{N} f(n) \qquad \text{and} \qquad I_N = \int_1^N f(x)\mathrm{d}x,$$

then $\sigma_N = s_N - I_N$. Hence

$$s_N = \sigma_N + I_N \qquad \text{and} \qquad I_N = s_N - \sigma_N.$$

Since σ_N converges as $N \to \infty$, it now follows from Proposition 19C that the convergence of one of s_N and I_N leads to the convergence of the other. ○

Problems for Chapter 20

1. Let $x_n = -1/n$ if 3 divides n, and $x_n = 1/n$ otherwise. Show, by considering the partial sums s_{3N}, that the series $\displaystyle\sum_{n=1}^{\infty} x_n$ diverges.

2. For each of the following, use the Comparison test to determine whether the series is convergent:
   a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{1/2}}{n^2+3}$
   b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+5n-3}$
   c) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^3}$
   d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+7n+3}{2n^5+3}$

3. For each of the following, use the Ratio test to determine whether the series is convergent:
   a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$
   b) $\displaystyle\sum_{n=1}^{\infty} \frac{(3n)!(2n)!}{(4n)!n!}$
   c) $\displaystyle\sum_{n=1}^{\infty} \frac{(3n)!(2n)!}{(4n)!n!}3^n$
   d) $\displaystyle\sum_{n=1}^{\infty} \frac{(3n)!(2n)!}{(4n)!n!}2^n$

4. For each of the following, use the Alternating series test to show that the series is convergent:
   a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n(\sqrt{n+1}-\sqrt{n})$
   b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2-6n+10}$

5. For each of the following, determine whether the series is convergent:
   a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{n\pi}{2}\right)$
   b) $\displaystyle\sum_{n=1}^{\infty} (n!)^{1/n}$
   c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^2}+\frac{3}{n}\right)$
   d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^3}+\frac{(-1)^n}{n}\right)$
   e) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{(n!)^2}{(2n)!}3^n+\frac{n^{3/2}}{4n^3+1}\right)$
   f) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{(2n)!}{(n!)^2}-\frac{1}{n^2}\right)$

6. Use the Ratio test and the fact that $\displaystyle\left(1+\frac{1}{n}\right)^n \to \mathrm{e}$ as $n\to\infty$ to show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is convergent.

7. Find real sequences x_n and y_n such that $x_n - y_n \to 0$ as $n\to\infty$, the series $\displaystyle\sum_{n=1}^{\infty} x_n$ diverges, but the series $\displaystyle\sum_{n=1}^{\infty} y_n$ converges.

8. For every $n\in\mathbb{N}$, let $\displaystyle a_n = \frac{1}{n^{1/2}} + \frac{(-1)^n}{n}$.
   a) Show that $a_n \ge 0$ for every $n\in\mathbb{N}$.
   b) Show that $a_n \to 0$ as $n\to\infty$.
   c) Explain why the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}a_n$ is divergent.

   [Remark: This shows that we cannot omit the condition that a_n is decreasing in the hypothesis of the Alternating series test.]

9. For each of the following, determine all the values $a \in \mathbb{R}$ for which the series $\displaystyle\sum_{n=1}^{\infty} x_n$ converges:

a) $x_n = \dfrac{\cos na}{n^2}$ b) $x_n = a^{n^2}$ c) $x_n = n!a^n$ d) $x_n = n!a^{n!}$

Harder Problems for Chapter 20

10. Suppose that $x_n \geq 0$ and $y_n \geq 0$ for every $n \in \mathbb{N}$. Suppose further that $x_n/y_n \to 1$ as $n \to \infty$. Show that $\displaystyle\sum_{n=1}^{\infty} x_n$ and $\displaystyle\sum_{n=1}^{\infty} y_n$ either both converge or both diverge.

11. Suppose that $\displaystyle\sum_{n=1}^{\infty} x_n$ and $\displaystyle\sum_{n=1}^{\infty} y_n$ are both convergent series with positive terms. Show that $\displaystyle\sum_{n=1}^{\infty} x_n y_n$ converges. Discuss the case when x_n and y_n can take negative values.