FIRST YEAR CALCULUS

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Chapter 21

POWER SERIES

21.1. Introduction

Let $x \in \mathbb{R}$. In this chapter, we shall study series of the type

$$\sum_{n=0}^{\infty} a_n x^n \quad \text{where } a_0, a_1, a_2, \ldots \in \mathbb{R}, \tag{1}$$

known commonly as power series. Our discussion will still be valid if the variable x and the coefficients $a_0, a_1, a_2, \ldots$ take complex values. However, as in Chapter 20, we shall restrict our discussion to real series.

Example 21.1.1. The exponential series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges absolutely for every $x \in \mathbb{R}$. To see this, note first of all that the result is obvious if $x = 0$. If $x \neq 0$, we apply the Ratio test, and note that

$$\left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = \frac{|x|}{n+1} \to 0$$

as $n \to \infty$.

EXAMPLE 21.1.2. The logarithmic series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

converges absolutely for every $x \in \mathbb{R}$ satisfying $|x| < 1$ and diverges for every $x \in \mathbb{R}$ satisfying $|x| > 1$. To see this, note first of all that the result is obvious if $x = 0$. If $x \neq 0$, we apply the Ratio test, and note that

$$\left|\frac{(-1)^n x^{n+1}/(n+1)}{(-1)^{n-1}x^n/n}\right| = \frac{n|x|}{n+1} \to |x|$$

as $n \to \infty$.

EXAMPLE 21.1.3. The series

$$\sum_{n=1}^{\infty} n!x^n$$

diverges for every non-zero $x \in \mathbb{R}$. To see this, we use Proposition 20D, and note that for any fixed $x \neq 0$, the sequence $n!x^n$ does not converge to 0 as $n \to \infty$.

We shall establish in Section 21.4 the following two important results.

PROPOSITION 21A. (CONVERGENCE THEOREM FOR POWER SERIES) *For a power series of the form* (1), *exactly one of the following holds:*
(a) The series converges absolutely for every $x \in \mathbb{R}$.
(b) There exists a positive real number R such that the series converges absolutely for every $x \in \mathbb{R}$ satisfying $|x| < R$ and diverges for every $x \in \mathbb{R}$ satisfying $|x| > R$.
(c) The series diverges for every non-zero $x \in \mathbb{R}$.

A crucial step in the proof of Proposition 21A is summarized by the result below.

PROPOSITION 21B. *Suppose that the series* (1) *converges for a particular value $x = x_0$. Then the series converges absolutely for every $x \in \mathbb{R}$ satisfying $|x| < |x_0|$.*

DEFINITION. The number R in Proposition 21A is called the radius of convergence of the series (1). We also say that the radius of convergence is 0 if case (c) occurs, and that the series (1) has infinite radius of convergence if case (a) occurs.

REMARKS. (1) Proposition 21A does not indicate whether the series is convergent if $|x| = R$.

(2) The Ratio test is a powerful tool for determining the radius of convergence of a power series.

EXAMPLE 21.1.4. The logarithmic series

$$\sum_{n=0}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

has radius of convergence 1. If $x = 1$, then the series converges by the Alternating series test. Note, however, that the convergence is not absolute. If $x = -1$, then the series clearly diverges, in view of Proposition 20F.

21.2. Taylor Series

We begin by stating the following generalized version of the Mean value theorem.

PROPOSITION 21C. (TAYLOR'S THEOREM) *Suppose that $n \in \mathbb{N}$. Suppose further that a function $f(x)$ satisfies the following two conditions:*
(a) $f(x)$ and its first $(n-1)$ derivatives $f'(x), f''(x), \ldots, f^{(n-1)}(x)$ are continuous in the closed interval $[a, a+h]$; and
(b) the n-th derivative exists in the open interval $(a, a+h)$.
Then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a+\theta h),$$

where $\theta \in \mathbb{R}$ satisfies $0 < \theta < 1$.

Remark. Taylor's theorem is sometimes known as the Mean value theorem of the n-th order. Note that for $n = 1$, Taylor's theorem reduces to the Mean value theorem.

In Proposition 21C, we can write

$$f(a+h) = S_n + R_n,$$

where

$$S_n = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)$$

and

$$R_n = \frac{h^n}{n!}f^{(n)}(a+\theta h). \tag{2}$$

If $R_n \to 0$ as $n \to \infty$, then $S_n \to f(a+h)$ as $n \to \infty$. We therefore have the following series version of Taylor's theorem.

PROPOSITION 21D. (TAYLOR SERIES) *Suppose that a function $f(x)$ satisfies the following two conditions:*
(a) $f(x)$ and all its derivatives $f'(x), f''(x), \ldots$ are continuous in the closed interval $[a, a+h]$; and
(b) the sequence R_n defined by (2) converges to 0 as $n \to \infty$.
Then

$$f(a+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(a),$$

with the convention that $0! = 1$.

Remark. The Maclaurin series is the Taylor series in the special case $a = 0$. Under suitable conditions, we have

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0). \tag{3}$$

Example 21.2.1. Consider the function $f(x) = e^x$. Then $f(x)$ has derivatives of all order, all equal to e^x. Note that $f^{(n)}(0) = 1$ for every $n \in \mathbb{N} \cup \{0\}$. It follows that the Maclaurin series of the exponential function is given by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

This is the exponential series.

EXAMPLE 21.2.2. Consider the function $f(x) = \log(1+x)$. Then $f(x)$ has derivatives of all order near $x = 0$. Furthermore, for every $n \in \mathbb{N}$, we have

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

(try proving this by induction), so that $f^{(n)}(0) = (-1)^{n-1}(n-1)!$. Note also that $f(0) = 0$. It follows that the Maclaurin series for the function is given by

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.$$

This is the logarithmic series.

EXAMPLE 21.2.3. Consider the function $f(x) = (1+x)^\alpha$, where $\alpha \in \mathbb{R} \setminus \{0, 1, 2, 3, \ldots\}$. Then $f(x)$ has derivatives of all order near $x = 0$. Furthermore, for every $n \in \mathbb{N}$, we have

$$f^{(n)}(x) = \alpha(\alpha-1)\ldots(\alpha-n+1)(1+x)^{\alpha-n},$$

so that

$$f^{(n)}(0) = \alpha(\alpha-1)\ldots(\alpha-n+1).$$

Note also that $f(0) = 1$. It follows that the Maclaurin series for the function is given by

$$(1+x)^\alpha = \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} x^n.$$

This is the Extended binomial theorem.

EXAMPLE 21.2.4. Consider the function $f(x) = (1+x)^n$, where $n \in \mathbb{N}$. Then $f(x)$ has derivatives of all order near $x = 0$. Furthermore, for every $r = 1, \ldots, n$, we have

$$f^{(r)}(x) = n(n-1)\ldots(n-r+1)(1+x)^{n-r},$$

so that

$$f^{(r)}(0) = n(n-1)\ldots(n-r+1).$$

On the other hand, for every natural number $r > n$, we have $f^{(r)}(x) = 0$. Note also that $f(0) = 1$. It follows that the Maclaurin series for the function has zero coefficients beyond the term x^n and is given by

$$(1+x)^n = \sum_{r=0}^{n} \frac{n(n-1)\ldots(n-r+1)}{r!} x^r.$$

This is a special case of the Binomial theorem. For further discussion of the Binomial theorem, see Chapter 22.

21.3. Application to Differential Equations

In this section, we discuss a simple technique which will enable us to solve some simple differential equations of the form

$$a_0(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_2(x)y = 0, \qquad (4)$$

where $a_0(x), a_1(x), a_2(x)$ are polynomials.

The technique involves assuming a power series solution of the form

$$y = x^k \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+k}, \tag{5}$$

where the constants k and $c_0, c_1, c_2, c_3, \ldots$ are to be found. Note that there is no reason that the first term of the power series should be constant. However, we can now stipulate that $c_0 \neq 0$. Since the equation (4) is homogeneous, we may assume, without loss of generality, that $c_0 = 1$. From (5), we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sum_{n=0}^{\infty} (n+k) c_n x^{n+k-1} \tag{6}$$

and

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \sum_{n=0}^{\infty} (n+k)(n+k-1) c_n x^{n+k-2}. \tag{7}$$

We now substitute (5)–(7) into the left hand side of the equation (4) to obtain a series. Since the right hand side of (4) is zero, all the coefficients of this series must be equal to zero. We therefore obtain many equations involving the constants k and $c_0, c_1, c_2, c_3, \ldots$. The equation associated with the lowest power of x is called the indicial equation. It is a quadratic equation and yields two values of k. The other equations then provide systematically the values of $c_1, c_2, c_3, \ldots$, in terms of $c_0 = 1$ and each value of k.

We remark that if the two values of k differ by an integer, our technique may break down. Also, if the two values of k are identical, then only one series is produced. In this latter case, a second series involving logarithmic terms may be produced, but we shall not be concerned here with this case.

Example 21.3.1. Suppose that

$$4x \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2 \frac{\mathrm{d}y}{\mathrm{d}x} + y = 0. \tag{8}$$

Substituting (5)–(7) into (8), we obtain

$$4x \sum_{n=0}^{\infty} (n+k)(n+k-1) c_n x^{n+k-2} + 2 \sum_{n=0}^{\infty} (n+k) c_n x^{n+k-1} + \sum_{n=0}^{\infty} c_n x^{n+k} = 0. \tag{9}$$

Note that the left hand side of (9) is of the form

$$\begin{aligned} &\sum_{n=0}^{\infty} 4(n+k)(n+k-1) c_n x^{n+k-1} + \sum_{n=0}^{\infty} 2(n+k) c_n x^{n+k-1} + \sum_{n=1}^{\infty} c_{n-1} x^{n+k-1} \\ &\quad = (4k(k-1) + 2k) c_0 x^{k-1} + \sum_{n=1}^{\infty} ((4(n+k)(n+k-1) + 2(n+k)) c_n + c_{n-1}) x^{n+k-1} \\ &\quad = 2k(2k-1) c_0 x^{k-1} + \sum_{n=1}^{\infty} (2(n+k)(2n+2k-1) c_n + c_{n-1}) x^{n+k-1}. \end{aligned}$$

Equating all coefficients to zero, we have the indicial equation

$$k(2k-1) = 0;$$

also, for $n \geq 1$, we have

$$2(n+k)(2n+2k-1)c_n + c_{n-1} = 0. \tag{10}$$

The indicial equation has roots $k = 0$ and $k = 1/2$. Note also that (10) can be rewritten in the form

$$c_n = -\frac{c_{n-1}}{2(n+k)(2n+2k-1)}. \tag{11}$$

With $k = 0$, equation (11) becomes

$$c_n = -\frac{c_{n-1}}{2n(2n-1)}. \tag{12}$$

If we write $c_0 = 1$, then substituting $n = 1, 2, 3, \ldots$ successively into (12), we obtain

$$\begin{aligned} c_1 &= -\frac{1}{2} = -\frac{1}{2!}, \\ c_2 &= \frac{1}{2! \cdot 4 \cdot 3} = \frac{1}{4!}, \\ c_3 &= -\frac{1}{4! \cdot 6 \cdot 5} = -\frac{1}{6!}. \end{aligned}$$

It can be proved by induction that for every $n \in \mathbb{N}$,

$$c_n = \frac{(-1)^n}{(2n)!}.$$

This gives rise to a solution

$$y = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \ldots = \cos \sqrt{x}.$$

With $k = 1/2$, equation (11) becomes

$$c_n = -\frac{c_{n-1}}{2n(2n+1)}. \tag{13}$$

If we write $c_0 = 1$, then substituting $n = 1, 2, 3, \ldots$ successively into (13), we obtain

$$\begin{aligned} c_1 &= -\frac{1}{2 \cdot 3} = -\frac{1}{3!}, \\ c_2 &= \frac{1}{3! \cdot 4 \cdot 5} = \frac{1}{5!}, \\ c_3 &= -\frac{1}{5! \cdot 6 \cdot 7} = -\frac{1}{7!}. \end{aligned}$$

It can be proved by induction that for every $n \in \mathbb{N}$,

$$c_n = \frac{(-1)^n}{(2n+1)!}.$$

This gives rise to a solution

$$y = x^{1/2}\left(1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \ldots\right) = \sin \sqrt{x}.$$

We can now apply the Ratio test to check for convergence of the two series.

Example 21.3.2. Suppose that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - xy = 0. \tag{14}$$

Substituting (5)–(7) into (14), we obtain

$$\sum_{n=0}^{\infty}(n+k)(n+k-1)c_n x^{n+k-2} - x\sum_{n=0}^{\infty} c_n x^{n+k} = 0. \tag{15}$$

Note that the left hand side of (15) is of the form

$$\begin{aligned}&\sum_{n=0}^{\infty}(n+k)(n+k-1)c_n x^{n+k-2} - \sum_{n=3}^{\infty} c_{n-3} x^{n+k-2} \\ &= k(k-1)c_0x^{k-2} + (k+1)kc_1x^{k-1} + (k+2)(k+1)c_2x^k + \sum_{n=3}^{\infty}((n+k)(n+k-1)c_n - c_{n-3})x^{n+k-2}.\end{aligned}$$

Equating all coefficients to zero, we have the indicial equation

$$k(k-1) = 0,$$

and

$$(k+1)kc_1 = 0 \qquad \text{and} \qquad (k+2)(k+1)c_2 = 0;$$

also, for $n \geq 3$, we have

$$(n+k)(n+k-1)c_n - c_{n-3} = 0. \tag{16}$$

The indicial equation has roots $k = 0$ and $k = 1$. Note also that (16) can be rewritten in the form

$$c_n = \frac{c_{n-3}}{(n+k)(n+k-1)}, \tag{17}$$

and that $c_1 = c_2 = 0$. With $k = 0$, equation (17) becomes

$$c_n = \frac{c_{n-3}}{(n-1)n}. \tag{18}$$

It follows that $c_n = 0$ unless n is a multiple of 3. If we write $c_0 = 1$, then substituting $n = 3, 6, 9, \ldots$ successively into (18), we obtain

$$\begin{aligned}c_3 &= \frac{1}{2\cdot 3}, \\ c_6 &= \frac{1}{2\cdot 3\cdot 5\cdot 6}, \\ c_9 &= \frac{1}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9},\end{aligned}$$

and so on. This gives rise to a solution

$$y = 1 + \frac{x^3}{2\cdot 3} + \frac{x^6}{2\cdot 3\cdot 5\cdot 6} + \frac{x^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} + \ldots.$$

With $k = 1$, equation (17) becomes

$$c_n = \frac{c_{n-3}}{n(n+1)}. \tag{19}$$

It follows that $c_n = 0$ unless n is a multiple of 3. If we write $c_0 = 1$, then substituting $n = 3, 6, 9, \ldots$ successively into (19), we obtain

$$c_3 = \frac{1}{3 \cdot 4}, \quad c_6 = \frac{1}{3 \cdot 4 \cdot 6 \cdot 7}, \quad c_9 = \frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10},$$

and so on. This gives rise to a solution

$$y = x + \frac{x^4}{3 \cdot 4} + \frac{x^7}{3 \cdot 4 \cdot 6 \cdot 7} + \frac{x^{10}}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} + \ldots.$$

21.4. Further Discussion

We begin by establishing the Convergence theorem for power series.

PROOF OF PROPOSITION 21B. Suppose that

$$\sum_{n=0}^{\infty} a_n x_0^n$$

converges. Then it follows from Proposition 20C that $a_n x_0^n \to 0$ as $n \to \infty$. Recall that any convergent sequence is bounded, so that there exists $M \in \mathbb{R}$ such that $|a_n x_0^n| \leq M$ for every $n \in \mathbb{N} \cup \{0\}$. For every $x \in \mathbb{R}$ satisfying $|x| < |x_0|$, we have

$$|a_n x^n| \leq M \left| \frac{x}{x_0} \right|^n$$

for every $n \in \mathbb{N} \cup \{0\}$, so that the series (1) converges absolutely by comparing with the convergent geometric series

$$\sum_{n=0}^{\infty} \left| \frac{x}{x_0} \right|^n,$$

noting that $|x/x_0| < 1$. ◯

PROOF OF PROPOSITION 21A. Consider the set

$$S = \{x \geq 0 : \text{the series (1) converges}\}.$$

Clearly S contains the number 0. On the other hand, in view of Proposition 21B, S must be an interval with lower endpoint 0. Exactly one of the following three cases applies.

(a) If $S = [0, +\infty)$, then for every $x \in \mathbb{R}$, we can choose $x_0 \in S$ such that $|x| < x_0$. Since the series converges at x_0, it follows from Proposition 21B that the series converges absolutely at x.

(b) Suppose that $S = [0, R)$ or $S = [0, R]$ for some positive number R. For every $x \in \mathbb{R}$ satisfying $|x| < R$, we can choose $x_0 \in S$ such that $|x| < x_0$. Since the series converges at x_0, it follows from Proposition 21B that the series converges absolutely at x. On the other hand, for every $x \in \mathbb{R}$ satisfying $|x| > R$, we can choose $x_0 > R$ such that $|x| > x_0$. If the series converges at x, then it follows from Proposition 21B that the series converges absolutely at x_0, a contradiction. Hence the series must diverge at x.

(c) If $S = \{0\}$, then for every non-zero $x \in \mathbb{R}$, we can choose $x_0 > 0$ such that $|x| > x_0$. If the series converges at x, then it follows from Proposition 21B that the series converges absolutely at x_0, a contradiction. Hence the series must diverge at x. ◯

We complete this chapter by establishing Taylor's theorem.

PROOF OF PROPOSITION 21C. For every $t \in [0, h]$, write

$$g(t) = f(a+t) - f(a) - tf'(a) - \ldots - \frac{t^{n-1}}{(n-1)!} f^{(n-1)}(a) - \frac{t^n}{n!} C, \tag{20}$$

where we shall choose C to ensure that $g(h) = 0$. It is easy to check that

$$g(0) = g'(0) = \ldots = g^{(n-1)}(0) = 0.$$

We now proceed to use Rolle's theorem n times. Since $g(0) = g(h) = 0$, there exists $h_1 \in (0, h)$ such that $g'(h_1) = 0$. Since $g'(0) = g'(h_1) = 0$, there exists $h_2 \in (0, h_1)$ such that $g''(h_2) = 0$, and so on. Finally, since $g^{(n-1)}(0) = g^{(n-1)}(h_{n-1}) = 0$, there exists $h_n \in (0, h_{n-1})$ such that $g^{(n)}(h_n) = 0$. Clearly $0 < h_n < h$, and so $h_n = \theta h$ for some $\theta \in \mathbb{R}$ satisfying $0 < \theta < 1$. Observe now that

$$g^{(n)}(t) = f^{(n)}(a+t) - C.$$

It follows that $C = f^{(n)}(a + \theta h)$. The result follows on substituting this into (20), letting $t = h$ and noting that $g(h) = 0$. ◯

Problems for Chapter 21

1. Find the radius of convergence of each of the following power series:

   a) $\displaystyle\sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^3} x^n$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2} x^n$

   c) $\displaystyle\sum_{n=0}^{\infty} a_n x^n$, where $a_n = 1$ when n is a perfect square and $a_n = 0$ otherwise

2. Find all the terms up to and including x^3 in the Taylor expansion of each of the following functions:

   a) $f(x) = (x+1)\sin x$ b) $f(x) = \mathrm{e}^x \cos x$ c) $f(x) = \tan x$

3. a) Find the Maclaurin expansion of the functions $\sin x$ and $\cos x$.

   b) Replacing x by $\mathrm{i}x$ in Example 21.2.1, we obtain

$$\mathrm{e}^{\mathrm{i}x} = \sum_{n=0}^{\infty} \frac{(\mathrm{i}x)^n}{n!}.$$

   Use this and your result in part (a) to show that $\mathrm{e}^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x$.

4. Apply the technique discussed in Section 21.3 to find the general series solution for each of the following differential equations:

   a) $\displaystyle x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} + xy = 0$ b) $\displaystyle (1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0$

   c) $\displaystyle (1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0$ d) $\displaystyle (1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 12y = 0$

Harder Problems for Chapter 21

5. Suppose that $3 \le a_n \le 597$. Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$. Explain carefully each step of your argument.