

FIRST YEAR CALCULUS

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Chapter 22

THE BINOMIAL THEOREM

22.1. Finite Binomial Expansions

In many instances, one needs to study expressions like

$$(a + b)^n,$$

where $n \in \mathbb{N} \cup \{0\}$. Let us first of all look at a few small values of n . It is not difficult to see that

$$\begin{aligned}(a + b)^0 &= 1, \\(a + b)^1 &= a + b, \\(a + b)^2 &= a^2 + 2ab + b^2, \\(a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3, \\(a + b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4,\end{aligned}$$

and so on. We can display the coefficients in the form of the Pascal triangle below:

$$\begin{array}{cccccc} & & & 1 & & & \\ & & & & 1 & & \\ & & 1 & & 2 & & 1 \\ & 1 & & 3 & & 3 & & 1 \\ 1 & & 4 & & 6 & & 4 & & 1\end{array} \tag{1}$$

Of course, there is no reason to stop at $n = 4$. If we go on indefinitely, then for each $n \in \mathbb{N} \cup \{0\}$, we can write

$$(a + b)^n = c_{n,0}a^n + c_{n,1}a^{n-1}b + \dots + c_{n,n-1}ab^{n-1} + c_{n,n}b^n,$$

where the coefficients give rise to the row

$$c_{n,0} \quad c_{n,1} \quad \dots \quad c_{n,n-1} \quad c_{n,n}$$

in the Pascal triangle. However, what are the values of these coefficients?

To find the values of these coefficients, we first make two observations.

(1) In the Pascal triangle (1), every entry is the sum of the two entries immediately above it. For example,

$$\begin{array}{ccccccc} & & & & 1 & & & \\ & & & & & 1 & & \\ & & & 1 & & 2 & & 1 \\ & & 1 & & 2 & & 1 & \\ & 1 & & 3 & & 3 & & 1 \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

highlights the fact $c_{3,0} + c_{3,1} = c_{4,1}$. So is it true that

$$c_{n,r-1} + c_{n,r} = c_{n+1,r}$$

whenever $n \in \mathbb{N}$ and $1 \leq r \leq n$?

(2) Every row in the Pascal triangle (1) starts and ends with the entry 1. So is it true that

$$c_{n,0} = c_{n,n} = 1$$

whenever $n \in \mathbb{N} \cup \{0\}$?

The answer to these two questions are given by the following result.

PROPOSITION 22A. (BINOMIAL THEOREM) For every $n \in \mathbb{N} \cup \{0\}$, we have

$$(a + b)^n = c_{n,0}a^n + c_{n,1}a^{n-1}b + \dots + c_{n,n-1}ab^{n-1} + c_{n,n}b^n, \quad (2)$$

where, for every $r = 0, 1, \dots, n$, we have

$$c_{n,r} = \frac{n(n-1)\dots(n-r+1)}{r!}, \quad (3)$$

with the convention that $0! = 1$ and that the expression $n(n-1)\dots(n-r+1)$ represents 1 when $r = 0$.

PROOF. We shall prove this result by induction on n . Suppose that for every $n \in \mathbb{N} \cup \{0\}$ and every $r = 0, 1, \dots, n$, the term $c_{n,r}$ is given by (3). Note that $c_{0,0} = c_{1,0} = c_{1,1} = 1$, so that (2) holds when $n = 0$ and $n = 1$. Suppose now that for fixed n , we have

$$(a + b)^n = c_{n,0}a^n + c_{n,1}a^{n-1}b + \dots + c_{n,n-1}ab^{n-1} + c_{n,n}b^n.$$

Then

$$\begin{aligned} (a + b)^{n+1} &= (a + b)(a + b)^n = (a + b)(c_{n,0}a^n + c_{n,1}a^{n-1}b + \dots + c_{n,n-1}ab^{n-1} + c_{n,n}b^n) \\ &= (c_{n,0}a^{n+1} + c_{n,1}a^n b + \dots + c_{n,n-1}a^2 b^{n-1} + c_{n,n}ab^n) \\ &\quad + (c_{n,0}a^n b + c_{n,1}a^{n-1}b^2 + \dots + c_{n,n-1}ab^n + c_{n,n}b^{n+1}) \\ &= c_{n,0}a^{n+1} + (c_{n,0} + c_{n,1})a^n b + (c_{n,1} + c_{n,2})a^{n-1}b^2 + \dots + (c_{n,n-1} + c_{n,n})ab^n + c_{n,n}b^{n+1}. \end{aligned} \quad (4)$$

Note now that

$$c_{n,0} = c_{n+1,0} \quad \text{and} \quad c_{n,n} = c_{n+1,n+1}. \tag{5}$$

On the other hand, if $1 \leq r \leq n$, then

$$\begin{aligned} c_{n,r-1} + c_{n,r} &= \frac{n(n-1)\dots(n-r+2)}{(r-1)!} + \frac{n(n-1)\dots(n-r+1)}{r!} \\ &= \frac{n(n-1)\dots(n-r+2)}{(r-1)!} \left(1 + \frac{n-r+1}{r}\right) \\ &= \frac{n(n-1)\dots(n-r+2)}{(r-1)!} \frac{n+1}{r} \\ &= \frac{(n+1)n(n-1)\dots(n+1-r+1)}{r!} = c_{n+1,r}. \end{aligned} \tag{6}$$

Combining (4)–(6), we conclude that

$$(a+b)^{n+1} = c_{n+1,0}a^{n+1} + c_{n+1,1}a^nb + \dots + c_{n+1,n}ab^n + c_{n+1,n+1}b^{n+1}.$$

The result now follows from the Principle of induction. \circ

REMARK. We usually write

$$\binom{n}{r} = \frac{n(n-1)\dots(n-r+1)}{r!},$$

so that the Binomial theorem becomes

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \dots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n.$$

In fact, the binomial coefficient

$$\binom{n}{r} \tag{7}$$

represents the number of ways of choosing r objects from a collection of n objects. The reason that (7) is the coefficient for $a^{n-r}b^r$ in the expansion of $(a+b)^n$ is as follows: Since

$$(a+b)^n = \underbrace{(a+b)\dots(a+b)}_n,$$

it follows that from these n copies of $(a+b)$, we need to pick a exactly $(n-r)$ times and pick b exactly r times and multiply in order to get a term $a^{n-r}b^r$. It follows that the coefficient for $a^{n-r}b^r$ is the number of different ways that we can pick a exactly $(n-r)$ times and pick b exactly r times, and this is the binomial coefficient (7).

REMARKS. (1) For every $n \in \mathbb{N}$ and every $r = 1, \dots, n$, we have

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}.$$

(2) Letting $a = b = 1$ in the Binomial theorem, we see that for every $n \in \mathbb{N} \cup \{0\}$, we have

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n-1} + \binom{n}{n} = 2^n.$$

22.2. Infinite Binomial Expansions

Sometimes, we need to study expressions like

$$(1+x)^\alpha,$$

where $\alpha \in \mathbb{R}$ is not a non-negative integer. We can write down a series expression for the function as follows. However, we need to be careful about convergence of the series.

PROPOSITION 22B. (EXTENDED BINOMIAL THEOREM) *Suppose that $\alpha \in \mathbb{R}$. Then for every $x \in \mathbb{R}$ satisfying $|x| < 1$, we have*

$$(1+x)^\alpha = \sum_{r=0}^{\infty} \binom{\alpha}{r} x^r,$$

where for every $r = 0, 1, 2, \dots$, the extended binomial coefficient is given by

$$\binom{\alpha}{r} = \frac{\alpha(\alpha-1)\dots(\alpha-r+1)}{r!}.$$

Proposition 22B can be demonstrated using Taylor series; see Section 21.2. Note also that Proposition 22B reduces to the Binomial theorem for $a = 1$ and $b = x$ when α is a non-negative integer.

The special case when α is a negative integer is somewhat special, as we can calculate the extended binomial coefficients rather easily.

PROPOSITION 22C. *Suppose that $m \in \mathbb{N}$. Then for every $r = 0, 1, 2, \dots$, we have*

$$\binom{-m}{r} = (-1)^r \binom{m+r-1}{r}.$$

PROOF. We have

$$\begin{aligned} \binom{-m}{r} &= \frac{-m(-m-1)\dots(-m-r+1)}{r!} = (-1)^r \frac{m(m+1)\dots(m+r-1)}{r!} \\ &= (-1)^r \frac{(m+r-1)(m+r-2)\dots(m+r-1-r+1)}{r!} = (-1)^r \binom{m+r-1}{r} \end{aligned}$$

as required. \circ

PROBLEMS FOR CHAPTER 22

1. Find the first four terms of the series for $(1 + 2x)^{-3}$.
2. What is the coefficient of x^4 in the series expansion of $(1 + x + x^2)^{-4}$?