LINEAR ALGEBRA

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Chapter 4

VECTORS

4.1. Introduction

A vector is an object which has magnitude and direction.

EXAMPLE 4.1.1. We may be travelling north-east at 50 kph. In this case, the direction of the velocity is north-east and the magnitude of the velocity is 50 kph. We can describe our velocity in kph as

$$\left(\frac{50}{\sqrt{2}}, \frac{50}{\sqrt{2}}\right),$$

where the first coordinate describes the speed with which we are moving east and the second coordinate describes the speed with which we are moving north.

EXAMPLE 4.1.2. An object in the sky may be 100 metres away in the south-east direction 45 degrees upwards. In this case, the direction of its position is south-eastand 45 degrees upwards and the magnitude of its distance is 100 metres. We can describe the position of the object in metres as

$$\left(50, -50, \frac{100}{\sqrt{2}}\right),$$

where the first coordinate describes the distance east, the second coordinate describes the distance north and the third coordinate describes the distance up.

The purpose of this chapter is to study some relationship between algebra and geometry. We shall first study some algebra which is motivated by geometric considerations. We then use the algebra later to better understand some problems in geometry.

4.2. Vectors in $\mathbb{R}^2$

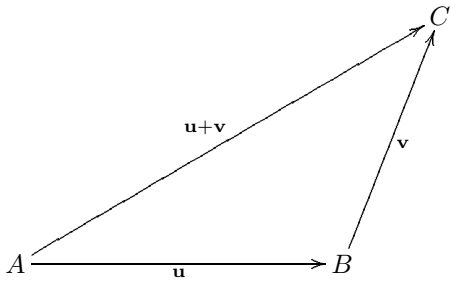
A vector on the plane $\mathbb{R}^2$ can be described as an ordered pair $\mathbf{u} = (u_1, u_2)$, where $u_1, u_2 \in \mathbb{R}$.

DEFINITION. Two vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ in $\mathbb{R}^2$ are said to be equal, denoted by $\mathbf{u} = \mathbf{v}$, if $u_1 = v_1$ and $u_2 = v_2$.

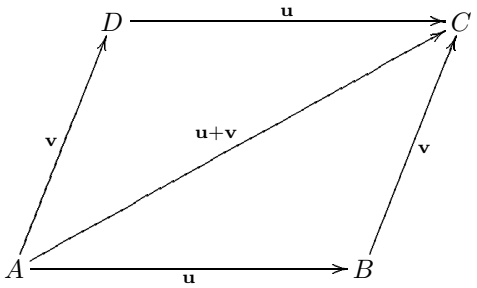
DEFINITION. For any two vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ in $\mathbb{R}^2$, we define their sum to be

$$\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2).$$

Geometrically, if we represent the two vectors $\mathbf{u}$ and $\mathbf{v}$ by $\overrightarrow{AB}$ and $\overrightarrow{BC}$ respectively, then the sum $\mathbf{u} + \mathbf{v}$ is represented by $\overrightarrow{AC}$ as shown in the diagram below:



The next diagram demonstrates geometrically that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$:



PROPOSITION 4A. (VECTOR ADDITION)
(a) For every $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, we have $\mathbf{u} + \mathbf{v} \in \mathbb{R}^2$.
(b) For every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$, we have $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
(c) For every $\mathbf{u} \in \mathbb{R}^2$, we have $\mathbf{u} + \mathbf{0} = \mathbf{u}$, where $\mathbf{0} = (0, 0) \in \mathbb{R}^2$.
(d) For every $\mathbf{u} \in \mathbb{R}^2$, there exists $\mathbf{v} \in \mathbb{R}^2$ such that $\mathbf{u} + \mathbf{v} = \mathbf{0}$.
(e) For every $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, we have $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

PROOF. Write $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$, where $u_1, u_2, v_1, v_2, w_1, w_2 \in \mathbb{R}$. To check part (a), simply note that $u_1 + v_1, u_2 + v_2 \in \mathbb{R}$. To check part (b), note that

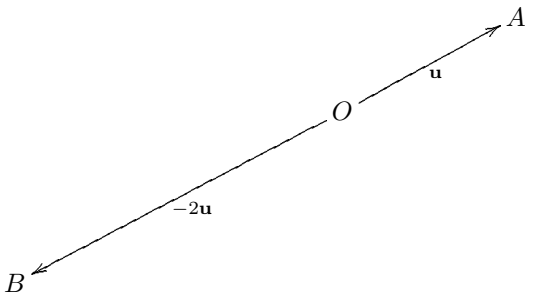
$$\begin{aligned}\mathbf{u} + (\mathbf{v} + \mathbf{w}) &= (u_1, u_2) + (v_1 + w_1, v_2 + w_2) = (u_1 + (v_1 + w_1), u_2 + (v_2 + w_2)) \\ &= ((u_1 + v_1) + w_1, (u_2 + v_2) + w_2) = (u_1 + v_1, u_2 + v_2) + (w_1, w_2) \\ &= (\mathbf{u} + \mathbf{v}) + \mathbf{w}.\end{aligned}$$

Part (c) is trivial. Next, if $\mathbf{v} = (-u_1, -u_2)$, then $\mathbf{u} + \mathbf{v} = \mathbf{0}$, giving part (d). To check part (e), note that $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2) = (v_1 + u_1, v_2 + u_2) = \mathbf{v} + \mathbf{u}$. ◯

DEFINITION. For any vector $\mathbf{u} = (u_1, u_2)$ in $\mathbb{R}^2$ and any scalar $c \in \mathbb{R}$, we define the scalar multiple to be

$$c\mathbf{u} = c(u_1, u_2) = (cu_1, cu_2).$$

EXAMPLE 4.2.1. Suppose that $\mathbf{u} = (2, 1)$. Then $-2\mathbf{u} = (-4, 2)$. Geometrically, if we represent the two vectors $\mathbf{u}$ and $-2\mathbf{u}$ by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively, then we have the diagram below:



PROPOSITION 4B. (SCALAR MULTIPLICATION)
(a) For every $c \in \mathbb{R}$ and $\mathbf{u} \in \mathbb{R}^2$, we have $c\mathbf{u} \in \mathbb{R}^2$.
(b) For every $c \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, we have $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
(c) For every $a, b \in \mathbb{R}$ and $\mathbf{u} \in \mathbb{R}^2$, we have $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.
(d) For every $a, b \in \mathbb{R}$ and $\mathbf{u} \in \mathbb{R}^2$, we have $(ab)\mathbf{u} = a(b\mathbf{u})$.
(e) For every $\mathbf{u} \in \mathbb{R}^2$, we have $1\mathbf{u} = \mathbf{u}$.

PROOF. Write $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, where $u_1, u_2, v_1, v_2 \in \mathbb{R}$. To check part (a), simply note that $cu_1, cu_2 \in \mathbb{R}$. To check part (b), note that

$$\begin{aligned} c(\mathbf{u} + \mathbf{v}) &= c(u_1 + v_1, u_2 + v_2) = (c(u_1 + v_1), c(u_2 + v_2)) \\ &= (cu_1 + cv_1, cu_2 + cv_2) = (cu_1, cu_2) + (cv_1, cv_2) = c\mathbf{u} + c\mathbf{v}. \end{aligned}$$

To check part (c), note that

$$\begin{aligned} (a + b)\mathbf{u} &= ((a + b)u_1, (a + b)u_2) = (au_1 + bu_1, au_2 + bu_2) \\ &= (au_1, au_2) + (bu_1, bu_2) = a\mathbf{u} + b\mathbf{u}. \end{aligned}$$

To check part (d), note that

$$(ab)\mathbf{u} = ((ab)u_1, (ab)u_2) = (a(bu_1), a(bu_2)) = a(bu_1, bu_2) = a(b\mathbf{u}).$$

Finally, to check part (e), note that $1\mathbf{u} = (1u_1, 1u_2) = (u_1, u_2) = \mathbf{u}$. ◯

DEFINITION. For any vector $\mathbf{u} = (u_1, u_2)$ in $\mathbb{R}^2$, we define the norm of $\mathbf{u}$ to be the non-negative real number

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2}.$$

REMARKS. (1) The norm of a vector is simply its magnitude or length. The definition follows from the famous theorem of Pythagoras.

(2) Suppose that $P(u_1, u_2)$ and $Q(v_1, v_2)$ are two points on the plane $\mathbb{R}^2$. To calculate the distance $d(P, Q)$ between the two points, we can first find a vector from P to Q. This is given by $(v_1 - u_1, v_2 - u_2)$. The distance $d(P, Q)$ is then the norm of this vector, so that

$$d(P, Q) = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2}.$$

(3) It is not difficult to see that for any vector $\mathbf{u} \in \mathbb{R}^2$ and any scalar $c \in \mathbb{R}$, we have $\|c\mathbf{u}\| = |c|\|\mathbf{u}\|$.

DEFINITION. Any vector $\mathbf{u} \in \mathbb{R}^2$ satisfying $\|\mathbf{u}\| = 1$ is called a unit vector.

EXAMPLE 4.2.2. The vector $(3, 4)$ has norm 5.

EXAMPLE 4.2.3. The distance between the points $(6, 3)$ and $(9, 7)$ is $\sqrt{(9-6)^2 + (7-3)^2} = 5$.

EXAMPLE 4.2.4. The vectors $(1, 0)$ and $(0, -1)$ are unit vectors in $\mathbb{R}^2$.

EXAMPLE 4.2.5. The unit vector in the direction of the vector $(1, 1)$ is $(1/\sqrt{2}, 1/\sqrt{2})$.

EXAMPLE 4.2.6. In fact, all unit vectors in $\mathbb{R}^2$ are of the form $(\cos\theta, \sin\theta)$, where $\theta \in \mathbb{R}$.

Quite often, we may want to find the angle between two vectors. The scalar product of the two vectors then comes in handy. We shall define the scalar product in two ways, one in terms of the angle between the two vectors and the other not in terms of this angle, and show that the two definitions are in fact equivalent.

DEFINITION. Suppose that $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are vectors in $\mathbb{R}^2$, and that $\theta \in [0, \pi]$ represents the angle between them. We define the scalar product $\mathbf{u} \cdot \mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ by

$$\mathbf{u} \cdot \mathbf{v} = \begin{cases} \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta & \text{if } \mathbf{u} \neq \mathbf{0} \text{ and } \mathbf{v} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{u} = \mathbf{0} \text{ or } \mathbf{v} = \mathbf{0}. \end{cases} \tag{1}$$

Alternatively, we write

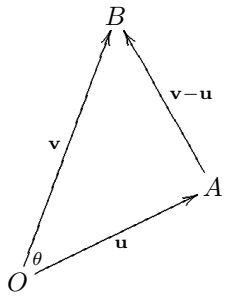
$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2. \tag{2}$$

The definitions (1) and (2) are clearly equivalent if $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$. On the other hand, we have the following result.

PROPOSITION 4C. *Suppose that $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are non-zero vectors in $\mathbb{R}^2$, and that $\theta \in [0, \pi]$ represents the angle between them. Then*

$$\|\mathbf{u}\|\|\mathbf{v}\| \cos\theta = u_1 v_1 + u_2 v_2.$$

PROOF. Geometrically, if we represent the two vectors $\mathbf{u}$ and $\mathbf{v}$ by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively, then the difference $\mathbf{v} - \mathbf{u}$ is represented by $\overrightarrow{AB}$ as shown in the diagram below:



By the Law of cosines, we have

$$\overline{AB}^2 = \overline{OA}^2 + \overline{OB}^2 - 2\overline{OA}\,\overline{OB}\cos\theta;$$

in other words, we have

$$\|\mathbf{v}-\mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta,$$

so that

$$\begin{aligned}\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta &= \tfrac{1}{2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v}-\mathbf{u}\|^2)\\ &= \tfrac{1}{2}(u_1^2+u_2^2+v_1^2+v_2^2-(v_1-u_1)^2-(v_2-u_2)^2)\\ &= u_1v_1+u_2v_2\end{aligned}$$

as required. ◯

REMARKS. (1) We say that two non-zero vectors in $\mathbb{R}^2$ are orthogonal if the angle between them is $\pi/2$. It follows immediately from the definition of the scalar product that two non-zero vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ are orthogonal if and only if $\mathbf{u}\cdot\mathbf{v} = 0$.

(2) We can calculate the scalar product of any two non-zero vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ by the formula (2) and then use the formula (1) to calculate the angle between $\mathbf{u}$ and $\mathbf{v}$.

EXAMPLE 4.2.7. Suppose that $\mathbf{u} = (\sqrt{3}, 1)$ and $\mathbf{v} = (\sqrt{3}, 3)$. Then by the formula (2), we have

$$\mathbf{u}\cdot\mathbf{v} = 3+3 = 6.$$

Note now that

$$\|\mathbf{u}\| = 2 \qquad \text{and} \qquad \|\mathbf{v}\| = 2\sqrt{3}.$$

It follows from the formula (1) that

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{6}{4\sqrt{3}} = \frac{\sqrt{3}}{2},$$

so that $\theta = \pi/6$.

EXAMPLE 4.2.8. Suppose that $\mathbf{u} = (\sqrt{3}, 1)$ and $\mathbf{v} = (-\sqrt{3}, 3)$. Then by the formula (2), we have $\mathbf{u}\cdot\mathbf{v} = 0$. It follows that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

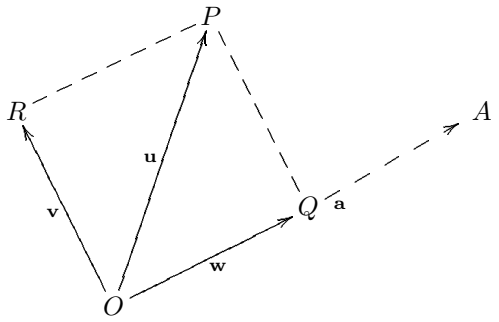
PROPOSITION 4D. (SCALAR PRODUCT) *Suppose that* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ *and* $c \in \mathbb{R}$. *Then*
(a) $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$;
(b) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = (\mathbf{u}\cdot\mathbf{v}) + (\mathbf{u}\cdot\mathbf{w})$;
(c) $c(\mathbf{u}\cdot\mathbf{v}) = (c\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(c\mathbf{v})$;
(d) $\mathbf{u}\cdot\mathbf{u} \geqslant 0$; *and*
(e) $\mathbf{u}\cdot\mathbf{u} = 0$ *if and only if* $\mathbf{u} = \mathbf{0}$.

PROOF. Write $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$, where $u_1, u_2, v_1, v_2, w_1, w_2 \in \mathbb{R}$. Part (a) is trivial. To check part (b), note that

$$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = u_1(v_1+w_1) + u_2(v_2+w_2) = (u_1v_1+u_2v_2) + (u_1w_1+u_2w_2) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}.$$

Part (c) is rather simple. To check parts (d) and (e), note that $\mathbf{u}\cdot\mathbf{u} = u_1^2 + u_2^2 \geqslant 0$, and that equality holds precisely when $u_1 = u_2 = 0$. ◯

Consider the diagram below:



(3)

Here we represent the two vectors $\mathbf{a}$ and $\mathbf{u}$ by $\overrightarrow{OA}$ and $\overrightarrow{OP}$ respectively. If we project the vector $\mathbf{u}$ on to the line $\overline{OA}$, then the image of the projection is the vector $\mathbf{w}$, represented by $\overrightarrow{OQ}$. On the other hand, if we project the vector $\mathbf{u}$ on to a line perpendicular to the line $\overline{OA}$, then the image of the projection is the vector $\mathbf{v}$, represented by $\overrightarrow{OR}$.

DEFINITION. In the notation of the diagram (3), the vector $\mathbf{w}$ is called the orthogonal projection of the vector $\mathbf{u}$ on the vector $\mathbf{a}$, and denoted by $\mathbf{w} = \mathrm{proj}_{\mathbf{a}}\mathbf{u}$.

PROPOSITION 4E. (ORTHOGONAL PROJECTION) *Suppose that* $\mathbf{u}, \mathbf{a} \in \mathbb{R}^2$. *Then*

$$\mathrm{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}.$$

REMARK. Note that the component of $\mathbf{u}$ orthogonal to $\mathbf{a}$, represented by $\overrightarrow{OR}$ in the diagram (3), is

$$\mathbf{u} - \mathrm{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}.$$

PROOF OF PROPOSITION 4E. Note that $\mathbf{w} = k\mathbf{a}$ for some $k \in \mathbb{R}$. It clearly suffices to prove that

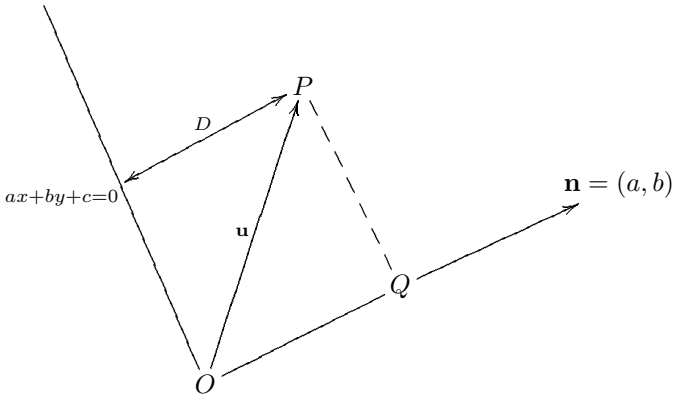
$$k = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}.$$

It is easy to see that the vectors $\mathbf{u} - \mathbf{w}$ and $\mathbf{a}$ are orthogonal. It follows that the scalar product $(\mathbf{u} - \mathbf{w}) \cdot \mathbf{a} = 0$. In other words, $(\mathbf{u} - k\mathbf{a}) \cdot \mathbf{a} = 0$. Hence

$$k = \frac{\mathbf{u} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}$$

as required. ◯

To end this section, we shall apply our knowledge gained so far to find a formula that gives the perpendicular distance of a point (x_0, y_0) from a line $ax + by + c = 0$. Consider the diagram below:



Suppose that (x_1, y_1) is any arbitrary point O on the line $ax + by + c = 0$. For any other point (x, y) on the line $ax + by + c = 0$, the vector $(x - x_1, y - y_1)$ is parallel to the line. On the other hand,

$$(a, b) \cdot (x - x_1, y - y_1) = (ax + by) - (ax_1 + by_1) = -c + c = 0,$$

so that the vector $\mathbf{n} = (a, b)$, in the direction $\overrightarrow{OQ}$, is perpendicular to the line $ax + by + c = 0$. Suppose next that the point (x_0, y_0) is represented by the point P in the diagram. Then the vector $\mathbf{u} = (x_0 - x_1, y_0 - y_1)$ is represented by $\overrightarrow{OP}$, and $\overrightarrow{OQ}$ represents the orthogonal projection $\mathrm{proj}_{\mathbf{n}}\mathbf{u}$ of $\mathbf{u}$ on the vector $\mathbf{n}$. Clearly the perpendicular distance D of the point (x_0, y_0) from the line $ax + by + c = 0$ satisfies

$$D = \|\mathrm{proj}_{\mathbf{n}}\mathbf{u}\| = \left\| \frac{\mathbf{u} \cdot \mathbf{n}}{\|\mathbf{n}\|^2} \mathbf{n} \right\| = \frac{|(x_0 - x_1, y_0 - y_1) \cdot (a, b)|}{\sqrt{a^2 + b^2}} = \frac{|ax_0 + by_0 - ax_1 - by_1|}{\sqrt{a^2 + b^2}} = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

We have proved the following result.

PROPOSITION 4F. *The perpendicular distance D of a point (x_0, y_0) from a line $ax + by + c = 0$ is given by*

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

EXAMPLE 4.2.9. The perpendicular distance D of the point $(5, 7)$ from the line $2x - 3y + 5 = 0$ is given by

$$D = \frac{|10 - 21 + 5|}{\sqrt{4 + 9}} = \frac{6}{\sqrt{13}}.$$

4.3. Vectors in $\mathbb{R}^3$

In this section, we consider the same problems as in Section 4.2, but in 3-space $\mathbb{R}^3$. Any reader who feels confident may skip this section.

A vector on the plane $\mathbb{R}^3$ can be described as an ordered triple $\mathbf{u} = (u_1, u_2, u_3)$, where $u_1, u_2, u_3 \in \mathbb{R}$.

DEFINITION. Two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ are said to be equal, denoted by $\mathbf{u} = \mathbf{v}$, if $u_1 = v_1$, $u_2 = v_2$ and $u_3 = v_3$.

DEFINITION. For any two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$, we define their sum to be

$$\mathbf{u} + \mathbf{v} = (u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1 + v_1, u_2 + v_2, u_3 + v_3).$$

DEFINITION. For any vector $\mathbf{u} = (u_1, u_2, u_3)$ in $\mathbb{R}^3$ and any scalar $c \in \mathbb{R}$, we define the scalar multiple to be

$$c\mathbf{u} = c(u_1, u_2, u_3) = (cu_1, cu_2, cu_3).$$

The following two results are the analogues of Propositions 4A and 4B. The proofs are essentially similar.

PROPOSITION 4A'. (VECTOR ADDITION)
(a) For every $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$*, we have* $\mathbf{u} + \mathbf{v} \in \mathbb{R}^3$*.*
(b) For every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$*, we have* $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$*.*
(c) For every $\mathbf{u} \in \mathbb{R}^3$*, we have* $\mathbf{u} + \mathbf{0} = \mathbf{u}$*, where* $\mathbf{0} = (0, 0, 0) \in \mathbb{R}^3$*.*
(d) For every $\mathbf{u} \in \mathbb{R}^3$*, there exists* $\mathbf{v} \in \mathbb{R}^3$ *such that* $\mathbf{u} + \mathbf{v} = \mathbf{0}$*.*
(e) For every $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$*, we have* $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$*.*

PROPOSITION 4B'. (SCALAR MULTIPLICATION)
(a) For every $c \in \mathbb{R}$ *and* $\mathbf{u} \in \mathbb{R}^3$*, we have* $c\mathbf{u} \in \mathbb{R}^3$*.*
(b) For every $c \in \mathbb{R}$ *and* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$*, we have* $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$*.*
(c) For every $a, b \in \mathbb{R}$ *and* $\mathbf{u} \in \mathbb{R}^3$*, we have* $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$*.*
(d) For every $a, b \in \mathbb{R}$ *and* $\mathbf{u} \in \mathbb{R}^3$*, we have* $(ab)\mathbf{u} = a(b\mathbf{u})$*.*
(e) For every $\mathbf{u} \in \mathbb{R}^3$*, we have* $1\mathbf{u} = \mathbf{u}$*.*

DEFINITION. For any vector $\mathbf{u} = (u_1, u_2, u_3)$ in $\mathbb{R}^3$, we define the norm of $\mathbf{u}$ to be the non-negative real number

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}.$$

REMARKS. (1) Suppose that $P(u_1, u_2, u_3)$ and $Q(v_1, v_2, v_3)$ are two points in $\mathbb{R}^3$. To calculate the distance $d(P, Q)$ between the two points, we can first find a vector from P to Q. This is given by $(v_1 - u_1, v_2 - u_2, v_3 - u_3)$. The distance $d(P, Q)$ is then the norm of this vector, so that

$$d(P, Q) = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2}.$$

(2) It is not difficult to see that for any vector $\mathbf{u} \in \mathbb{R}^3$ and any scalar $c \in \mathbb{R}$, we have

$$\|c\mathbf{u}\| = |c|\|\mathbf{u}\|.$$

DEFINITION. Any vector $\mathbf{u} \in \mathbb{R}^3$ satisfying $\|\mathbf{u}\| = 1$ is called a unit vector.

EXAMPLE 4.3.1. The vector $(3, 4, 12)$ has norm 13.

EXAMPLE 4.3.2. The distance between the points $(6, 3, 12)$ and $(9, 7, 0)$ is 13.

EXAMPLE 4.3.3. The vectors $(1, 0, 0)$ and $(0, -1, 0)$ are unit vectors in $\mathbb{R}^3$.

EXAMPLE 4.3.4. The unit vector in the direction of the vector $(1, 0, 1)$ is $(1/\sqrt{2}, 0, 1/\sqrt{2})$.

The theory of scalar products can be extended to $\mathbb{R}^3$ is the natural way.

DEFINITION. Suppose that $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are vectors in $\mathbb{R}^3$, and that $\theta \in [0, \pi]$ represents the angle between them. We define the scalar product $\mathbf{u} \cdot \mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ by

$$\mathbf{u} \cdot \mathbf{v} = \begin{cases} \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta & \text{if } \mathbf{u} \neq \mathbf{0} \text{ and } \mathbf{v} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{u} = \mathbf{0} \text{ or } \mathbf{v} = \mathbf{0}. \end{cases} \tag{4}$$

Alternatively, we write

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3. \tag{5}$$

The definitions (4) and (5) are clearly equivalent if $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$. On the other hand, we have the following analogue of Proposition 4C. The proof is similar.

PROPOSITION 4C'. *Suppose that* $\mathbf{u} = (u_1, u_2, u_3)$ *and* $\mathbf{v} = (v_1, v_2, v_3)$ *are non-zero vectors in* $\mathbb{R}^3$*, and that* $\theta \in [0, \pi]$ *represents the angle between them. Then*

$$\|\mathbf{u}\|\|\mathbf{v}\| \cos\theta = u_1v_1 + u_2v_2 + u_3v_3.$$

REMARKS. (1) We say that two non-zero vectors in $\mathbb{R}^3$ are orthogonal if the angle between them is $\pi/2$. It follows immediately from the definition of the scalar productthat two non-zero vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ are orthogonal if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

(2) We can calculate the scalar product of any two non-zero vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ by the formula (5) and then use the formula (4) to calculate the angle between $\mathbf{u}$ and $\mathbf{v}$.

EXAMPLE 4.3.5. Suppose that $\mathbf{u} = (2, 0, 0)$ and $\mathbf{v} = (1, 1, \sqrt{2})$. Then by the formula (5), we have $\mathbf{u} \cdot \mathbf{v} = 2$. Note now that $\|\mathbf{u}\| = 2$ and $\|\mathbf{v}\| = 2$. It follows from the formula (4) that

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{2}{4} = \frac{1}{2},$$

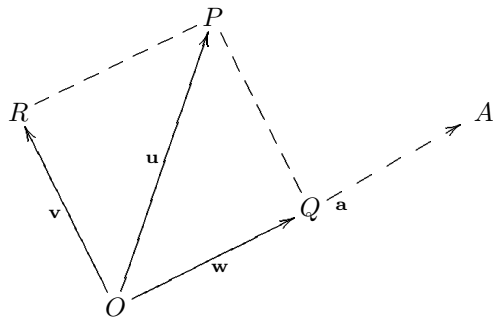
so that $\theta = \pi/3$.

EXAMPLE 4.3.6. Suppose that $\mathbf{u} = (2, 3, 5)$ and $\mathbf{v} = (1, 1, -1)$. Then by the formula (5), we have $\mathbf{u} \cdot \mathbf{v} = 0$. It follows that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

The following result is the analogue of Proposition 4D. The proof is similar.

PROPOSITION 4D'. (SCALAR PRODUCT) *Suppose that* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ *and* $c \in \mathbb{R}$*. Then*
(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$*;*
(b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$*;*
(c) $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$*;*
(d) $\mathbf{u} \cdot \mathbf{u} \geq 0$*; and*
(e) $\mathbf{u} \cdot \mathbf{u} = 0$ *if and only if* $\mathbf{u} = \mathbf{0}$*.*

Suppose now that $\mathbf{a}$ and $\mathbf{u}$ are two vectors in $\mathbb{R}^3$. Then since two vectors are always coplanar, we can draw the following diagram which represents the plane they lie on:



(6)

Note that this diagram is essentially the same as the diagram (3), the only difference being that while the diagram (3) shows the whole of $\mathbb{R}^2$, the diagram (6) only shows part of $\mathbb{R}^3$. As before, we represent the two vectors $\mathbf{a}$ and $\mathbf{u}$ by $\overrightarrow{OA}$ and $\overrightarrow{OP}$ respectively. If we project the vector $\mathbf{u}$ on to the line $\overline{OA}$, then the image of the projection is the vector $\mathbf{w}$, represented by $\overrightarrow{OQ}$. On the other hand, if we project the vector $\mathbf{u}$ on to a line perpendicular to the line $\overline{OA}$, then the image of the projection is the vector $\mathbf{v}$, represented by $\overrightarrow{OR}$.

DEFINITION. In the notation of the diagram (6), the vector $\mathbf{w}$ is called the orthogonal projection of the vector $\mathbf{u}$ on the vector $\mathbf{a}$, and denoted by $\mathbf{w} = \mathrm{proj}_{\mathbf{a}}\mathbf{u}$.

The following result is the analogue of Proposition 4E. The proof is similar.

PROPOSITION 4E'. (ORTHOGONAL PROJECTION) *Suppose that* $\mathbf{u}, \mathbf{a} \in \mathbb{R}^3$. *Then*

$$\mathrm{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}.$$

REMARK. Note that the component of $\mathbf{u}$ orthogonal to $\mathbf{a}$, represented by $\overrightarrow{OR}$ in the diagram (6), is

$$\mathbf{u} - \mathrm{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}.$$

4.4. Vector Products

In this section, we shall discuss a product of vectors unique to $\mathbb{R}^3$. The idea of vector products has wide applications in geometry, physics and engineering, and is motivated by the wish to find a vector that is perpendicular to two given vectors.

We shall use the right hand rule. In other words, if we hold the thumb on the right hand upwards and close the remaining four fingers, then the fingers point from the x-direction towards the y-direction, while the thumb points towards the z-direction. Alternatively, if we imagine Columbus had never lived and that the earth were flat, then taking the x-direction as east and the y-direction as north, then the z-direction is upwards!

We shall frequently use the three vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$ in $\mathbb{R}^3$.

DEFINITION. Suppose that $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are two vectors in $\mathbb{R}^3$. Then the vector product $\mathbf{u} \times \mathbf{v}$ is defined by the determinant

$$\mathbf{u} \times \mathbf{v} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}.$$

REMARKS. (1) Note that

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= -(\mathbf{j} \times \mathbf{i}) = \mathbf{k}, \\ \mathbf{j} \times \mathbf{k} &= -(\mathbf{k} \times \mathbf{j}) = \mathbf{i}, \\ \mathbf{k} \times \mathbf{i} &= -(\mathbf{i} \times \mathbf{k}) = \mathbf{j}. \end{aligned}$$

(2) Using cofactor expansion by row 1, we have

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \det \begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix} \mathbf{i} - \det \begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix} \mathbf{j} + \det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \mathbf{k} \\ &= \left(\det \begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix}, -\det \begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix}, \det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \right) \\ &= (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1). \end{aligned}$$

We shall first of all show that the vector product $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

PROPOSITION 4G. *Suppose that* $\mathbf{u} = (u_1, u_2, u_3)$ *and* $\mathbf{v} = (v_1, v_2, v_3)$ *are two vectors in* $\mathbb{R}^3$. *Then*
(a) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$*; and*
(b) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$.

PROOF. Note first of all that

$$\begin{aligned}\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) &= (u_1, u_2, u_3) \cdot \left(\det \begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix}, -\det \begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix}, \det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \right) \\ &= u_1 \det \begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix} - u_2 \det \begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix} + u_3 \det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} = \det \begin{pmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix},\end{aligned}$$

in view of cofactor expansion by row 1. On the other hand, clearly

$$\det \begin{pmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} = 0.$$

This proves part (a). The proof of part (b) is similar. ◯

EXAMPLE 4.4.1. Suppose that $\mathbf{u} = (1, -1, 2)$ and $\mathbf{v} = (3, 0, 2)$. Then

$$\mathbf{u} \times \mathbf{v} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 3 & 0 & 2 \end{pmatrix} = \left(\det \begin{pmatrix} -1 & 2 \\ 0 & 2 \end{pmatrix}, -\det \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}, \det \begin{pmatrix} 1 & -1 \\ 3 & 0 \end{pmatrix} \right) = (-2, 4, 3).$$

Note that $(1, -1, 2) \cdot (-2, 4, 3) = 0$ and $(3, 0, 2) \cdot (-2, 4, 3) = 0$.

PROPOSITION 4H. (VECTOR PRODUCT) *Suppose that* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ *and* $c \in \mathbb{R}$. *Then*
(a) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$*;*
(b) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$*;*
(c) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$*;*
(d) $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$*;*
(e) $\mathbf{u} \times \mathbf{0} = \mathbf{0}$*; and*
(f) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.

PROOF. Write $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$. To check part (a), note that

$$\det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} = -\det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{pmatrix}.$$

To check part (b), note that

$$\det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 + w_1 & v_2 + w_2 & v_3 + w_3 \end{pmatrix} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} + \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

Part (c) is similar. To check part (d), note that

$$c \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ cu_1 & cu_2 & cu_3 \\ v_1 & v_2 & v_3 \end{pmatrix} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ cv_1 & cv_2 & cv_3 \end{pmatrix}.$$

To check parts (e) and (f), note that

$$\mathbf{u} \times \mathbf{0} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ 0 & 0 & 0 \end{pmatrix} = 0 \qquad \text{and} \qquad \mathbf{u} \times \mathbf{u} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \end{pmatrix} = 0$$

as required. ◯

Next, we shall discuss an application of vector product to the evaluaton of the area of a parallelogram. To do this, we shall first establish the following result.

PROPOSITION 4J. *Suppose that* $\mathbf{u} = (u_1, u_2, u_3)$ *and* $\mathbf{v} = (v_1, v_2, v_3)$ *are non-zero vectors in* $\mathbb{R}^3$*, and that* $\theta \in [0, \pi]$ *represents the angle between them. Then*
(a) $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$*; and*
(b) $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$*.*

Proof. Note that

$$\|\mathbf{u} \times \mathbf{v}\|^2 = (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \tag{7}$$

and

$$\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 = (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2. \tag{8}$$

Part (a) follows on expanding the right hand sides of (7) and (8) and checking that they are equal. To prove part (b), recall that

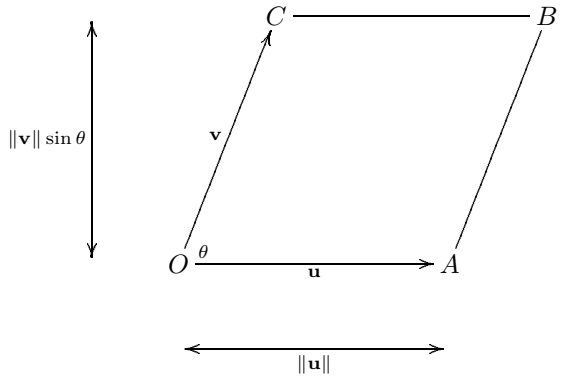
$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta.$$

Combining with part (a), we obtain

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \cos^2\theta = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \sin^2\theta.$$

Part (b) follows. ◯

Consider now a parallelogram with vertices O, A, B, C. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are represented by $\overrightarrow{OA}$ and $\overrightarrow{OC}$ respectively. If we imagine the side $\overline{OA}$ to represent the base of the parallelogram, so that the base has length $\|\mathbf{u}\|$, then the height of the the parallelogram is given by $\|\mathbf{v}\| \sin\theta$, as shown in the diagram below:



It follows from Proposition 4J that the area of the parallelogram is given by $\|\mathbf{u} \times \mathbf{v}\|$. We have proved the following result.

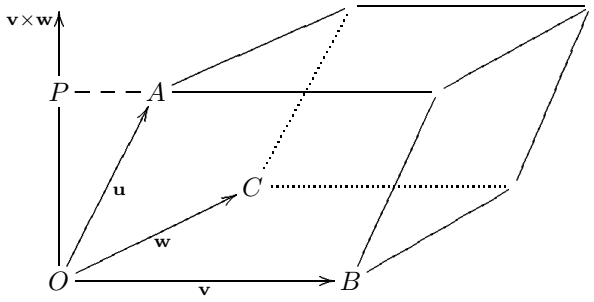
PROPOSITION 4K. *Suppose that* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$*. Then the parallelogram with* $\mathbf{u}$ *and* $\mathbf{v}$ *as two of its sides has area* $\|\mathbf{u} \times \mathbf{v}\|$*.*

We conclude this section by making a remark on the vector product $\mathbf{u} \times \mathbf{v}$ of two vectors in $\mathbb{R}^3$. Recall that the vector product is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. Furthermore, it can be shown that the direction of $\mathbf{u} \times \mathbf{v}$ satisfies the right hand rule, in the sense that if we hold the thumb on the right hand outwards and close the remaining four fingers, then the thumb points towards the $\mathbf{u} \times \mathbf{v}$-direction when the fingers point from the $\mathbf{u}$-direction towards the $\mathbf{v}$-direction. Also, we showed in Proposition 4J that

the magnitude of $\mathbf{u} \times \mathbf{v}$ depends only on the norm of $\mathbf{u}$ and $\mathbf{v}$ and the angle between the two vectors. It follows that the vector product is unchanged as long as we keep a right hand coordinate system. This is an important consideration in physics and engineering, where we may use different coordinate systems on the same problem.

4.5. Scalar Triple Products

Suppose that $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ do not all lie on the same plane. Consider the parallelepiped with $\mathbf{u}, \mathbf{v}, \mathbf{w}$ as three of its edges. We are interested in calculating the volume of this parallelepiped. Suppose that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are represented by $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ respectively. Consider the diagram below:



By Proposition 4K, the base of this parallelepiped, with O, B, C as three of the vertices, has area $\|\mathbf{v} \times \mathbf{w}\|$. Next, note that if $\overrightarrow{OP}$ is perpendicular to the base of the parallelepiped, then $\overrightarrow{OP}$ is in the direction of $\mathbf{v} \times \mathbf{w}$. If $\overrightarrow{PA}$ is perpendicular to $\overrightarrow{OP}$, then the height of the parallelepiped is equal to the norm of the orthogonal projection of $\mathbf{u}$ on $\mathbf{v} \times \mathbf{w}$. In other words, the parallelepiped has height

$$\|\mathrm{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}\| = \left\| \frac{\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})}{\|\mathbf{v} \times \mathbf{w}\|^2} (\mathbf{v} \times \mathbf{w}) \right\| = \frac{|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|}{\|\mathbf{v} \times \mathbf{w}\|}.$$

Hence the volume of the parallelepiped is given by

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|.$$

We have proved the following result.

PROPOSITION 4L. *Suppose that* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. *Then the parallelepiped with* $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *as three of its edges has volume* $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$.

Definition. Suppose that $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Then $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is called the scalar triple product of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

Remarks. (1) It follows immediately from Proposition 4L that three vectors in $\mathbb{R}^3$ are coplanar if and only if their scalar triple product is zero.

(2) Note that

$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= (u_1, u_2, u_3) \cdot \left(\det \begin{pmatrix} v_2 & v_3 \\ w_2 & w_3 \end{pmatrix}, -\det \begin{pmatrix} v_1 & v_3 \\ w_1 & w_3 \end{pmatrix}, \det \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix} \right) \\
&= u_1 \det \begin{pmatrix} v_2 & v_3 \\ w_2 & w_3 \end{pmatrix} - u_2 \det \begin{pmatrix} v_1 & v_3 \\ w_1 & w_3 \end{pmatrix} + u_3 \det \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix} \\
&= \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}, \qquad (9)
\end{aligned}$$

in view of cofactor expansion by row 1.

(3) It follows from identity (9) that

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}).$$

Note that each of the determinants can be obtained from the other two by twice interchanging two rows.

EXAMPLE 4.5.1. Suppose that $\mathbf{u} = (1, 0, 1)$, $\mathbf{v} = (2, 1, 3)$ and $\mathbf{w} = (0, 1, 1)$. Then

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{pmatrix} = 0,$$

so that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are coplanar.

EXAMPLE 4.5.2. The volume of the parallelepiped with $\mathbf{u} = (1, 0, 1)$, $\mathbf{v} = (2, 1, 4)$ and $\mathbf{w} = (0, 1, 1)$ as three of its edges is given by

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = \left| \det \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 4 \\ 0 & 1 & 1 \end{pmatrix} \right| = |-1| = 1.$$

4.6. Application to Geometry in $\mathbb{R}^3$

In this section, we shall study lines and planes in $\mathbb{R}^3$ by using our results on vectors in $\mathbb{R}^3$.

Consider first of all a plane in $\mathbb{R}^3$. Suppose that $(x_1, y_1, z_1) \in \mathbb{R}^3$ is a given point on this plane. Suppose further that $\mathbf{n} = (a, b, c)$ is a vector perpendicular to this plane. Then for any arbitrary point $(x, y, z) \in \mathbb{R}^3$ on this plane, the vector

$$(x, y, z) - (x_1, y_1, z_1) = (x - x_1, y - y_1, z - z_1)$$

joins one point on the plane to another point on the plane, and so must be parallel to the plane and hence perpendicular to $\mathbf{n} = (a, b, c)$. It follows that the scalar product

$$(a, b, c) \cdot (x - x_1, y - y_1, z - z_1) = 0,$$

and so

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0. \tag{10}$$

If we write $-d = ax_1 + by_1 + cz_1$, then (10) can be rewritten in the form

$$ax + by + cz + d = 0. \tag{11}$$

Equation (10) is usually called the point-normal form of the equation of a plane, while equation (11) is usually known as the general form of the equation of a plane.

EXAMPLE 4.6.1. Consider the plane through the point $(2, -5, 7)$ and perpendicular to the vector $(3, 5, -4)$. Here $(a, b, c) = (3, 5, -4)$ and $(x_1, y_1, z_1) = (2, -5, 7)$. The equation of the plane is given in point-normal form by $3(x-2)+5(y+5)-4(z-7) = 0$, and in general form by $3x+5y-4z+37 = 0$. Here $-d = 6 - 25 - 28 = -37$.

EXAMPLE 4.6.2. Consider the plane through the points $(1,1,1)$, $(2,2,0)$ and $(4,-6,2)$. Then the vectors

$$(2,2,0)-(1,1,1)=(1,1,-1) \qquad \text{and} \qquad (4,-6,2)-(1,1,1)=(3,-7,1)$$

join the point $(1,1,1)$ to the points $(2,2,0)$ and $(4,-6,2)$ respectively and are therefore parallel to the plane. It follows that the vector product

$$(1,1,-1)\times(3,-7,1)=(-6,-4,-10)$$

is perpendicular to the plane. The equation of the plane is then given by $-6(x-1)-4(y-1)-10(z-1)=0$, or $3x+2y+5z-10=0$.

Consider next a line in $\mathbb{R}^3$. Suppose that $(x_1,y_1,z_1)\in\mathbb{R}^3$ is a given point on this line. Suppose further that $\mathbf{n}=(a,b,c)$ is a vector parallel to this line. Then for any arbitrary point $(x,y,z)\in\mathbb{R}^3$ on this line, the vector

$$(x,y,z)-(x_1,y_1,z_1)=(x-x_1,y-y_1,z-z_1)$$

joins one point on the line to another point on the line, and so must be parallel to $\mathbf{n}=(a,b,c)$. It follows that there is some number $\lambda\in\mathbb{R}$ such that

$$(x-x_1,y-y_1,z-z_1)=\lambda(a,b,c),$$

so that

$$\begin{aligned} x&=x_1+a\lambda,\\ y&=y_1+b\lambda,\\ z&=z_1+c\lambda, \end{aligned} \tag{12}$$

where λ is called a parameter. Suppose further that a,b,c are all non-zero. Then, eliminating the parameter λ, we obtain

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}. \tag{13}$$

Equations (12) are usually called the parametric form of the equations of a line, while equations (13) are usually known as the symmetric form of the equations of a line.

EXAMPLE 4.6.3. Consider the line through the point $(2,-5,7)$ and parallel to the vector $(3,5,-4)$. Here $(a,b,c)=(3,5,-4)$ and $(x_1,y_1,z_1)=(2,-5,7)$. The equations of the line are given in parametric form by

$$\begin{aligned} x&=2+3\lambda,\\ y&=-5+5\lambda,\\ z&=7-4\lambda, \end{aligned}$$

and in symmetric form by

$$\frac{x-2}{3}=\frac{y+5}{5}=-\frac{z-7}{4}.$$

EXAMPLE 4.6.4. Consider the line through the points $(3, 0, 5)$ and $(7, 0, 8)$. Then a vector in the direction of the line is given by

$$(7, 0, 8) - (3, 0, 5) = (4, 0, 3).$$

The equation of the line is then given in parametric form by

$$\begin{aligned} x &= 3 + 4\lambda, \\ y &= 0, \\ z &= 5 + 3\lambda, \end{aligned}$$

and in symmetric form by

$$\frac{x-3}{4} = \frac{z-5}{3} \qquad \text{and} \qquad y = 0.$$

Consider the plane through three fixed points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3), not lying on the same line. Let (x, y, z) be a point on the plane. Then the vectors

$$\begin{aligned} (x, y, z) - (x_1, y_1, z_1) &= (x - x_1, y - y_1, z - z_1), \\ (x, y, z) - (x_2, y_2, z_2) &= (x - x_2, y - y_2, z - z_2), \\ (x, y, z) - (x_3, y_3, z_3) &= (x - x_3, y - y_3, z - z_3), \end{aligned}$$

each joining one point on the plane to another point on the plane, are all parallel to the plane. Using the vector product, we see that the vector

$$(x - x_2, y - y_2, z - z_2) \times (x - x_3, y - y_3, z - z_3)$$

is perpendicular to the plane, and so perpendicular to the vector $(x - x_1, y - y_1, z - z_1)$. It follows that the scalar triple product

$$(x - x_1, y - y_1, z - z_1) \cdot ((x - x_2, y - y_2, z - z_2) \times (x - x_3, y - y_3, z - z_3)) = 0;$$

in other words,

$$\det \begin{pmatrix} x - x_1 & y - y_1 & z - z_1 \\ x - x_2 & y - y_2 & z - z_2 \\ x - x_3 & y - y_3 & z - z_3 \end{pmatrix} = 0.$$

This is another technique to find the equation of a plane through three fixed points.

EXAMPLE 4.6.5. We return to the plane in Example 4.6.2, through the three points $(1, 1, 1)$, $(2, 2, 0)$ and $(4, -6, 2)$. The equation is given by

$$\det \begin{pmatrix} x - 1 & y - 1 & z - 1 \\ x - 2 & y - 2 & z - 0 \\ x - 4 & y + 6 & z - 2 \end{pmatrix} = 0.$$

The determinant on the left hand side is equal to $-6x - 4y - 10z + 20$. Hence the equation of the plane is given by $-6x - 4y - 10z + 20 = 0$, or $3x + 2y + 5z - 10 = 0$.

We observe that the calculation for the determinant above is not very pleasant. However, the technique can be improved in the following way by making less reference to the unknown point (x, y, z). Note that the vectors

$$\begin{aligned} (x, y, z) - (x_1, y_1, z_1) &= (x - x_1, y - y_1, z - z_1), \\ (x_2, y_2, z_2) - (x_1, y_1, z_1) &= (x_2 - x_1, y_2 - y_1, z_2 - z_1), \\ (x_3, y_3, z_3) - (x_1, y_1, z_1) &= (x_3 - x_1, y_3 - y_1, z_3 - z_1), \end{aligned}$$

each joining one point on the plane to another point on the plane, are all parallel to the plane. Using the vector product, we see that the vector

$$(x_2 - x_1, y_2 - y_1, z_2 - z_1) \times (x_3 - x_1, y_3 - y_1, z_3 - z_1)$$

is perpendicular to the plane, and so perpendicular to the vector $(x - x_1, y - y_1, z - z_1)$. It follows that the scalar triple product

$$(x - x_1, y - y_1, z - z_1) \cdot ((x_2 - x_1, y_2 - y_1, z_2 - z_1) \times (x_3 - x_1, y_3 - y_1, z_3 - z_1)) = 0;$$

in other words,

$$\det \begin{pmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{pmatrix} = 0.$$

EXAMPLE 4.6.6. We return to the plane in Examples 4.6.2 and 4.6.5, through the three points $(1, 1, 1)$, $(2, 2, 0)$ and $(4, -6, 2)$. The equation is given by

$$\det \begin{pmatrix} x - 1 & y - 1 & z - 1 \\ 2 - 1 & 2 - 1 & 0 - 1 \\ 4 - 1 & -6 - 1 & 2 - 1 \end{pmatrix} = 0.$$

The determinant on the left hand side is equal to

$$\det \begin{pmatrix} x - 1 & y - 1 & z - 1 \\ 1 & 1 & -1 \\ 3 & -7 & 1 \end{pmatrix} = -6(x - 1) - 4(y - 1) - 10(z - 1) = -6x - 4y - 10z + 20.$$

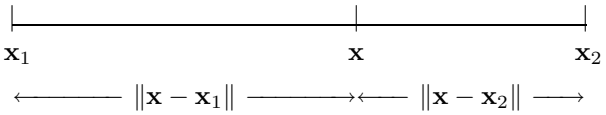
Hence the equation of the plane is given by $-6x - 4y - 10z + 20 = 0$, or $3x + 2y + 5z - 10 = 0$.

We next consider the problem of dividing a line segment in a given ratio. Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ are two given points in $\mathbb{R}^3$.

We wish to divide the line segment joining $\mathbf{x}_1$ and $\mathbf{x}_2$ internally in the ratio $\alpha_1 : \alpha_2$, where α_1 and α_2 are positive real numbers. In other words, we wish to find the point $\mathbf{x}$ on the line segment joining $\mathbf{x}_1$ and $\mathbf{x}_2$ such that

$$\frac{\|\mathbf{x} - \mathbf{x}_1\|}{\|\mathbf{x} - \mathbf{x}_2\|} = \frac{\alpha_1}{\alpha_2},$$

as shown in the diagram below:



Since $\mathbf{x} - \mathbf{x}_1$ and $\mathbf{x}_2 - \mathbf{x}$ are both in the same direction as $\mathbf{x}_2 - \mathbf{x}_1$, we must have

$$\alpha_2(\mathbf{x} - \mathbf{x}_1) = \alpha_1(\mathbf{x}_2 - \mathbf{x}), \qquad \text{or} \qquad \mathbf{x} = \frac{\alpha_1\mathbf{x}_2 + \alpha_2\mathbf{x}_1}{\alpha_1 + \alpha_2}.$$

We wish next to find the point $\mathbf{x}$ on the line joining $\mathbf{x}_1$ and $\mathbf{x}_2$, but not between $\mathbf{x}_1$ and $\mathbf{x}_2$, such that

$$\frac{\|\mathbf{x} - \mathbf{x}_1\|}{\|\mathbf{x} - \mathbf{x}_2\|} = \frac{\alpha_1}{\alpha_2},$$

where α_1 and α_2 are positive real numbers, as shown in the diagrams below for the cases $\alpha_1 < \alpha_2$ and $\alpha_1 > \alpha_2$ respectively:

Since $\mathbf{x} - \mathbf{x}_1$ and $\mathbf{x} - \mathbf{x}_2$ are in the same direction as each other, we must have

$$\alpha_2(\mathbf{x} - \mathbf{x}_1) = \alpha_1(\mathbf{x} - \mathbf{x}_2), \qquad \text{or} \qquad \mathbf{x} = \frac{\alpha_1\mathbf{x}_2 - \alpha_2\mathbf{x}_1}{\alpha_1 - \alpha_2}.$$

EXAMPLE 4.6.7. Let $\mathbf{x}_1 = (1, 2, 3)$ and $\mathbf{x}_2 = (7, 11, 6)$. The point

$$\mathbf{x} = \frac{2\mathbf{x}_2 + \mathbf{x}_1}{2 + 1} = \frac{2(7, 11, 6) + (1, 2, 3)}{3} = (5, 8, 5)$$

divides the line segment joining $(1, 2, 3)$ and $(7, 11, 6)$ internally in the ratio $2 : 1$, whereas the point

$$\mathbf{x} = \frac{4\mathbf{x}_2 - 2\mathbf{x}_1}{4 - 2} = \frac{4(7, 11, 6) - 2(1, 2, 3)}{2} = (13, 20, 9)$$

satisfies

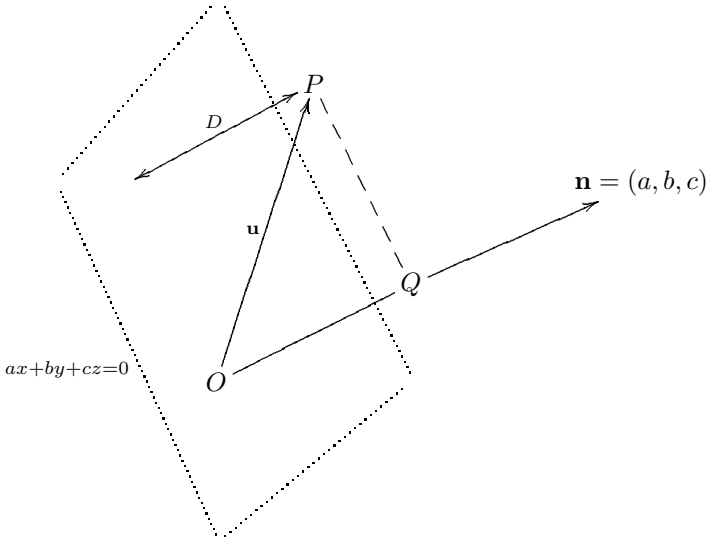
$$\frac{\|\mathbf{x} - \mathbf{x}_1\|}{\|\mathbf{x} - \mathbf{x}_2\|} = \frac{4}{2}.$$

Finally we turn our attention to the question of finding the distance of a plane from a given point. We shall prove the following analogue of Proposition 4F.

PROPOSITION 4F'. *The perpendicular distance D of a plane $ax + by + cz + d = 0$ from a point (x_0, y_0, z_0) is given by*

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

PROOF. Consider the following diagram:



Suppose that (x_1, x_2, x_3) is any arbitrary point O on the plane $ax+by+cz+d=0$. For any other point (x, y, z) on the plane $ax + by + cz + d = 0$, the vector $(x - x_1, y - y_1, z - z_1)$ is parallel to the plane. On the other hand,

$$(a, b, c) \cdot (x - x_1, y - y_1, z - z_1) = (ax + by + cz) - (ax_1 + by_1 + cz_1) = -d + d = 0,$$

so that the vector $\mathbf{n} = (a, b, c)$, in the direction $\overrightarrow{OQ}$, is perpendicular to the plane $ax + by + cz + d = 0$. Suppose next that the point (x_0, y_0, z_0) is represented by the point P in the diagram. Then the vector $\mathbf{u} = (x_0 - x_1, y_0 - y_1, z_0 - z_1)$ is represented by $\overrightarrow{OP}$, and $\overrightarrow{OQ}$ represents the orthogonal projection $\mathrm{proj}_{\mathbf{n}}\mathbf{u}$ of $\mathbf{u}$ on the vector $\mathbf{n}$. Clearly the perpendicular distance D of the point (x_0, y_0, z_0) from the plane $ax + by + cz + d = 0$ satisfies

$$\begin{aligned} D = \|\mathrm{proj}_{\mathbf{n}}\mathbf{u}\| &= \left\| \frac{\mathbf{u} \cdot \mathbf{n}}{\|\mathbf{n}\|^2} \mathbf{n} \right\| = \frac{|(x_0 - x_1, y_0 - y_1, z_0 - z_1) \cdot (a, b, c)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|ax_0 + by_0 + cz_0 - ax_1 - by_1 - cz_1|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

as required. ◯

A special case of Proposition 4F' is when $(x_0, y_0, z_0) = (0, 0, 0)$ is the origin. This show that the perpendicular distance of the plane $ax + by + cz + d = 0$ from the origin is

$$\frac{|d|}{\sqrt{a^2 + b^2 + c^2}}.$$

EXAMPLE 4.6.8. Consider the plane $3x + 5y - 4z + 37 = 0$. The distance of the point $(1, 2, 3)$ from the plane is

$$\frac{|3 + 10 - 12 + 37|}{\sqrt{9 + 25 + 16}} = \frac{38}{\sqrt{50}} = \frac{19\sqrt{2}}{5}.$$

The distance of the origin from the plane is

$$\frac{|37|}{\sqrt{9 + 25 + 16}} = \frac{37}{\sqrt{50}}.$$

EXAMPLE 4.6.9. Consider also the plane $3x+5y-4z-1 = 0$. Note that this plane is also perpendicular to the vector $(3, 5, -4)$ and is therefore parallel to the plane $3x+5y-4z+37 = 0$. It is therefore reasonable to find the perpendicular distance between these two parallel planes. Note that the perpendicular distance between the two planes is equal to the perpendicular distance of any point on $3x + 5y - 4z - 1 = 0$ from the plane $3x + 5y - 4z + 37 = 0$. Note now that $(1, 2, 3)$ lies on the plane $3x + 5y - 4z - 1 = 0$. It follows from Example 4.6.8 that the distance between the two planes is $19\sqrt{2}/5$.

4.7. Application to Mechanics

Let $\mathbf{u} = (u_x, u_y)$ denote a vector in $\mathbb{R}^2$, where the components u_x and u_y are functions of an independent variable t. Then the derivative of $\mathbf{u}$ with respect to t is given by

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \left(\frac{\mathrm{d}u_x}{\mathrm{d}t}, \frac{\mathrm{d}u_y}{\mathrm{d}t} \right).$$

EXAMPLE 4.7.1. When discussing planar particle motion, we often let $\mathbf{r} = (x, y)$ denote the position of a particle at time t. Then the components x and y are functions of t. The derivative

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}\right)$$

represents the velocity of the particle, and its derivative

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \left(\frac{\mathrm{d}^2x}{\mathrm{d}t^2}, \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\right)$$

represents the acceleration of the particle. We often write $r = \|\mathbf{r}\|$, $v = \|\mathbf{v}\|$ and $a = \|\mathbf{a}\|$.

Suppose that $\mathbf{w} = (w_x, w_y)$ is another vector in $\mathbb{R}^2$. Then it is not difficult to see that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u} \cdot \mathbf{w}) = \mathbf{u} \cdot \frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} \cdot \mathbf{w}. \tag{14}$$

EXAMPLE 4.7.2. Consider a particle moving at constant speed along a circular path centred at the origin. Then $r = \|\mathbf{r}\|$ is constant. More precisely, the position vector $\mathbf{r} = (x, y)$ satisfies $x^2 + y^2 = c_1$, where c_1 is a positive constant, so that

$$\mathbf{r} \cdot \mathbf{r} = (x, y) \cdot (x, y) = c_1. \tag{15}$$

On the other hand, $v = \|\mathbf{v}\|$ is constant. More precisely, the velocity vector

$$\mathbf{v} = \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}\right) \qquad \text{satisfies} \qquad \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = c_2,$$

where c_2 is a positive constant, so that

$$\mathbf{v} \cdot \mathbf{v} = \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}\right) \cdot \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}\right) = c_2. \tag{16}$$

Differentiating (15) and (16) with respect to t, and using the identity (14), we obtain respectively

$$\mathbf{r} \cdot \mathbf{v} = 0 \qquad \text{and} \qquad \mathbf{v} \cdot \mathbf{a} = 0. \tag{17}$$

Using the properties of the scalar product, we see that the equations in (17) show that the vector $\mathbf{v}$ is perpendicular to both vectors $\mathbf{r}$ and $\mathbf{a}$, and so $\mathbf{a}$ must be in the same direction as or the opposite direction to $\mathbf{r}$. Next, differentiating the first equation in (17), we obtain

$$\mathbf{r} \cdot \mathbf{a} + \mathbf{v} \cdot \mathbf{v} = 0, \qquad \text{or} \qquad \mathbf{r} \cdot \mathbf{a} = -v^2 < 0.$$

Let θ denote the angle between $\mathbf{a}$ and $\mathbf{r}$. Then $\theta = 0^\circ$ or $\theta = 180^\circ$. Since

$$\mathbf{r} \cdot \mathbf{a} = \|\mathbf{r}\|\|\mathbf{a}\| \cos\theta,$$

it follows that $\cos\theta < 0$, and so $\theta = 180^\circ$. We also obtain $ra = v^2$, so that $a = v^2/r$. This is a vector proof that for circular motion at constant speed, the acceleration is towards the centre of the circle and of magnitude v^2/r.

Let $\mathbf{u} = (u_x.u_y, u_z)$ denote a vector in $\mathbb{R}^3$, where the components u_x, u_y and u_z are functions of an independent variable t. Then the derivative of $\mathbf{u}$ with respect to t is given by

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \left(\frac{\mathrm{d}u_x}{\mathrm{d}t}, \frac{\mathrm{d}u_y}{\mathrm{d}t}, \frac{\mathrm{d}u_z}{\mathrm{d}t}\right).$$

Suppose that $\mathbf{w} = (w_x, w_y, w_z)$ is another vector in $\mathbb{R}^3$. Then it is not difficult to see that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u} \cdot \mathbf{w}) = \mathbf{u} \cdot \frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} \cdot \mathbf{w}. \tag{18}$$

EXAMPLE 4.7.3. When discussing particle motion in 3-dimensional space, we often let $\mathbf{r} = (x, y, z)$ denote the position of a particle at time t. Then the components x, y and z are functions of t. The derivative

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t} \right) = (\dot{x}, \dot{y}, \dot{z})$$

represents the velocity of the particle, and its derivative

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \left(\frac{\mathrm{d}^2x}{\mathrm{d}t^2}, \frac{\mathrm{d}^2y}{\mathrm{d}t^2}, \frac{\mathrm{d}^2z}{\mathrm{d}t^2} \right) = (\ddot{x}, \ddot{y}, \ddot{z})$$

represents the acceleration of the particle.

EXAMPLE 4.7.4. For a particle of mass m, the kinetic energy is given by

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \tfrac{1}{2}m(\dot{x}, \dot{y}, \dot{z}) \cdot (\dot{x}, \dot{y}, \dot{z}) = \tfrac{1}{2}m\mathbf{v} \cdot \mathbf{v}.$$

Using the identity (18), we have

$$\frac{\mathrm{d}T}{\mathrm{d}t} = m\mathbf{a} \cdot \mathbf{v} = \mathbf{F} \cdot \mathbf{v},$$

where $\mathbf{F} = m\mathbf{a}$ denotes the force. On the other hand, suppose that the potential energy is given by V. Using knowledge on functions of several real variables, we can show that

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial V}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial V}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right) \cdot \mathbf{v} = \nabla V \cdot \mathbf{v},$$

where

$$\nabla V = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right)$$

is called the gradient of V. The law of conservation of energy says that $T + V$ is constant, so that

$$\frac{\mathrm{d}T}{\mathrm{d}t} + \frac{\mathrm{d}V}{\mathrm{d}t} = (F + \nabla V) \cdot \mathbf{v} = 0$$

holds for all vectors $\mathbf{v}$, so that $\mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r})$ for all vectors $\mathbf{r}$.

EXAMPLE 4.7.5. If a force acts on a moving particle, then the work done is defined as the product of the distance moved and the magnitude of the force in the direction of motion. Suppose that a force $\mathbf{F}$ acts on a particle with displacement $\mathbf{r}$. Then the component of the force in the direction of the motion is given by $\mathbf{F} \cdot \mathbf{u}$, where

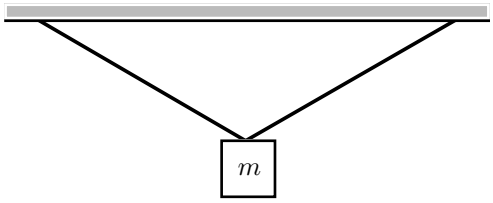
$$\mathbf{u} = \frac{\mathbf{r}}{\|\mathbf{r}\|}$$

is a unit vector in the direction of the vector $\mathbf{r}$. It follows that the work done is given by

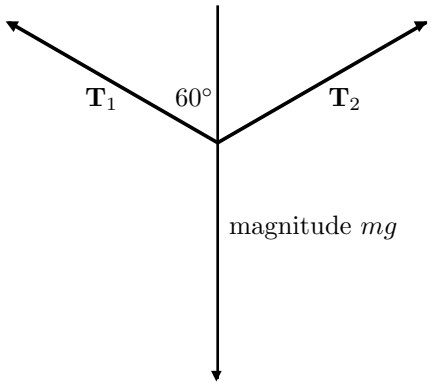
$$\|\mathbf{r}\| \left(\mathbf{F} \cdot \frac{\mathbf{r}}{\|\mathbf{r}\|} \right) = \mathbf{F} \cdot \mathbf{r}.$$

For instance, we see that the work done in moving a particle along a vector $\mathbf{r} = (3, -2, 4)$ with applied force $\mathbf{F} = (2, -1, 1)$ is $\mathbf{F} \cdot \mathbf{r} = (2, -1, 1) \cdot (3, -2, 4) = 12$.

EXAMPLE 4.7.6. We can also resolve a force into components. Consider a weight of mass m hanging from the ceiling on a rope as shown in the picture below:



Here the rope makes an angle of 60° with the vertical. We wish to find the tension T on the rope. To find this, note that the tension on the rope is a force, and we have the following picture of forces:



The force $\mathbf{T}_1$ has magnitude $\|\mathbf{T}_1\| = T$. Let $\mathbf{z}$ be a unit vector pointing vertically upwards. Using scalar products, we see that the component of the force $\mathbf{T}_1$ in the vertical direction is

$$\mathbf{T}_1 \cdot \mathbf{z} = \|\mathbf{T}_1\| \|\mathbf{z}\| \cos 60^\circ = \tfrac{1}{2} T.$$

Similarly, the force $\mathbf{T}_2$ has magnitude $\|\mathbf{T}_2\| = T$, and the component of it in the vertical direction is

$$\mathbf{T}_2 \cdot \mathbf{z} = \|\mathbf{T}_2\| \|\mathbf{z}\| \cos 60^\circ = \tfrac{1}{2} T.$$

Since the weight is stationary, he total force upwards on it is $\frac{1}{2}T + \frac{1}{2}T - mg = 0$. Hence $T = mg$.

Problems for Chapter 4

1. For each of the following pairs of vectors in $\mathbb{R}^2$, calculate $\mathbf{u}+3\mathbf{v}$, $\mathbf{u}\cdot\mathbf{v}$, $\|\mathbf{u}-\mathbf{v}\|$ and find the angle between $\mathbf{u}$ and $\mathbf{v}$:
   a) $\mathbf{u}=(1,1)$ and $\mathbf{v}=(-5,0)$ b) $\mathbf{u}=(1,2)$ and $\mathbf{v}=(2,1)$

2. For each of the following pairs of vectors in $\mathbb{R}^2$, calculate $2\mathbf{u}-5\mathbf{v}$, $\|\mathbf{u}-2\mathbf{v}\|$, $\mathbf{u}\cdot\mathbf{v}$ and the angle between $\mathbf{u}$ and $\mathbf{v}$ (to the nearest degree):
   a) $\mathbf{u}=(1,3)$ and $\mathbf{v}=(-2,1)$ b) $\mathbf{u}=(2,0)$ and $\mathbf{v}=(-1,2)$

3. For the two vectors $\mathbf{u}=(2,3)$ and $\mathbf{v}=(5,1)$ in the 2-dimensional euclidean space $\mathbb{R}^2$, determine each of the following:
   a) $\mathbf{u}-\mathbf{v}$ b) $\|\mathbf{u}\|$
   c) $\mathbf{u}\cdot(\mathbf{u}-\mathbf{v})$ d) the angle between $\mathbf{u}$ and $\mathbf{u}-\mathbf{v}$

4. For each of the following pairs of vectors in $\mathbb{R}^3$, calculate $\mathbf{u}+3\mathbf{v}$, $\mathbf{u}\cdot\mathbf{v}$, $\|\mathbf{u}-\mathbf{v}\|$, find the angle between $\mathbf{u}$ and $\mathbf{v}$, and find a unit vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$:
   a) $\mathbf{u}=(1,1,1)$ and $\mathbf{v}=(-5,0,5)$ b) $\mathbf{u}=(1,2,3)$ and $\mathbf{v}=(3,2,1)$

5. Find vectors $\mathbf{v}$ and $\mathbf{w}$ such that $\mathbf{v}$ is parallel to $(1,2,3)$, $\mathbf{v}+\mathbf{w}=(7,3,5)$ and $\mathbf{w}$ is orthogonal to $(1,2,3)$.

6. Let $ABCD$ be a quadrilateral. Show that the quadrilateral obtained by joining the midpoints of adjacent sides of $ABCD$ is a parallelogram.
   [Hint: Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ be vectors representing the four sides of $ABCD$.]

7. Suppose that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^3$ such that the scalar triple porduct $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})\neq 0$. Let

$$\mathbf{u}'=\frac{\mathbf{v}\times\mathbf{w}}{\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})},\qquad \mathbf{v}'=\frac{\mathbf{w}\times\mathbf{u}}{\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})},\qquad \mathbf{w}'=\frac{\mathbf{u}\times\mathbf{v}}{\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})}.$$

   a) Show that $\mathbf{u}'\cdot\mathbf{u}=1$.
   b) Show that $\mathbf{u}'\cdot\mathbf{v}=\mathbf{u}'\cdot\mathbf{w}=0$.
   c) Use the properties of the scalar triple product to find $\mathbf{v}'\cdot\mathbf{v}$ and $\mathbf{w}'\cdot\mathbf{w}$, as well as $\mathbf{v}'\cdot\mathbf{u}$, $\mathbf{v}'\cdot\mathbf{w}$, $\mathbf{w}'\cdot\mathbf{u}$ and $\mathbf{w}'\cdot\mathbf{v}$.

8. Suppose that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}'$, $\mathbf{v}'$ and $\mathbf{w}'$ are vectors in $\mathbb{R}^3$ such that $\mathbf{u}'\cdot\mathbf{u}=\mathbf{v}'\cdot\mathbf{v}=\mathbf{w}'\cdot\mathbf{w}=1$ and $\mathbf{u}'\cdot\mathbf{v}=\mathbf{u}'\cdot\mathbf{w}=\mathbf{v}'\cdot\mathbf{u}=\mathbf{v}'\cdot\mathbf{w}=\mathbf{w}'\cdot\mathbf{u}=\mathbf{w}'\cdot\mathbf{v}=0$. Show that if $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})\neq 0$, then

$$\mathbf{u}'=\frac{\mathbf{v}\times\mathbf{w}}{\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})},\qquad \mathbf{v}'=\frac{\mathbf{w}\times\mathbf{u}}{\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})},\qquad \mathbf{w}'=\frac{\mathbf{u}\times\mathbf{v}}{\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})}.$$

9. Suppose that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^3$.
   a) Show that $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})=(\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$.
   b) Deduce that $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}=(\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{v}\cdot\mathbf{w})\mathbf{u}$.

10. Consider the three points $P(2,3,1)$, $Q(4,2,5)$ and $R(1,6,-3)$.
    a) Find the equation of the line through P and Q.
    b) Find the equation of the plane perpendicular to the line in part (a) and passing through R.
    c) Find the distance between R and the line in part (a).
    d) Find the area of the parallelogram with the three points as vertices.
    e) Find the equation of the plane through the three points.
    f) Find the distance of the origin $(0,0,0)$ from the plane in part (e).
    g) Are the planes in parts (b) and (e) perpendicular? Justify your assertion.

11. Consider the points $(1, 2, 3)$, $(0, 2, 4)$ and $(2, 1, 3)$ in $\mathbb{R}^3$.
    a) Find the area of a parallelogram with these points as three of its vertices.
    b) Find the perpendicular distance between $(1, 2, 3)$ and the line passing through $(0, 2, 4)$ and $(2, 1, 3)$.

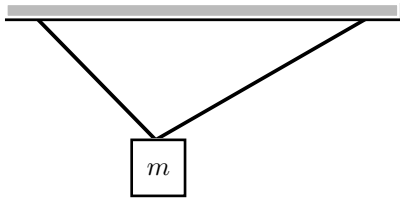
12. Consider the points $(1, 2, 3)$, $(0, 2, 4)$ and $(2, 1, 3)$ in $\mathbb{R}^3$.
    a) Find a vector perpendicular to the plane containing these points.
    b) Find the equation of this plane and its perpendicular distance from the origin.
    c) Find the equation of the line perpendicular to this plane and passing through the point $(3, 6, 9)$.

13. Find the equation of the plane through the points $(1, 2, -3)$, $(2, -3, 4)$ and $(-3, 1, 2)$.

14. Find the equation of the plane through the points $(2, -1, 1)$, $(3, 2, -1)$ and $(-1, 3, 2)$.

15. Find the volume of a parallelepiped with the points $(1, 2, 3)$, $(0, 2, 4)$, $(2, 1, 3)$ and $(3, 6, 9)$ as four of its vertices.

16. Consider a weight of mass m hanging from the ceiling supported by two ropes as shown in the picture below:



    Here the rope on the left makes an angle of 45° with the vertical, while the rope on the right makes an angle of 60° with the vertical. Find the tension on the two ropes.