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DAVENPORT'S THEOREM IN THE THEORY OF IRREGULARITIES OF POINT DISTRIBUTION

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We study distributions $\mathcal{D}_N$ of N points in the unit square U^2 with minimal order of L_2-discrepancy $\mathcal{L}_2[\mathcal{D}_N] < C(\log N)^{1/2}$, where the constant C is independent of N. We present an approach that uses Walsh functions and can be generalized to higher dimensions. Bibliography: 19 *titles.*

1. Introduction

Assume that $\mathcal{A}_N$ is a distribution of $N > 1$ points, not necessarily distinct, in the unit square $U^2 = [0,1)^2$. The L_2-discrepancy $\mathcal{L}_2[\mathcal{A}_N]$ is defined by the formula

$$\mathcal{L}_2[\mathcal{A}_N] = \left(\int_{U^2} |\mathcal{L}[\mathcal{A}_N;Y]|^2\,dY\right)^{1/2},$$

where, for every $Y = (y_1, y_2) \in U^2$, the local discrepancy $\mathcal{L}[\mathcal{A}_N;Y]$ is given by

$$\mathcal{L}[\mathcal{A}_N;Y] = \#(\mathcal{A}_N \cap B_Y) - Ny_1y_2. \tag{1.1}$$

Here, $B_Y = [0,y_1) \times [0,y_2) \subseteq U^2$ is a rectangle of area y_1y_2, while $\#(\mathcal{S})$ denotes the number of points of a set $\mathcal{S}$, counted with multiplicity.

The following results are classical.

Theorem 1 (Roth [13]). *There exists a positive absolute constant c such that, for any distribution $\mathcal{A}_N$ of N points in the unit square U^2, we have*

$$\mathcal{L}_2[\mathcal{A}_N] > c(\log N)^{1/2}.$$

Theorem 2 (Davenport [4]). *There exists a positive absolute constant C such that, for every natural number $N > 1$, there exist distributions $\mathcal{B}_N$ of N points in the unit square U^2 such that*

$$\mathcal{L}_2[\mathcal{B}_N] < C(\log N)^{1/2}.$$

Theorem 1 can easily be generalized to higher dimensions, with a lower bound of the form $c_n(\log N)^{\frac{1}{2}(n-1)}$ for an n-dimensional analog. Theorem 2 can also be extended to higher dimensions, as is shown by Roth [16], with an upper bound of the form $C_n(\log N)^{\frac{1}{2}(n-1)}$ for an n-dimensional analog. However, Roth's argument involves considerable extra difficulties.

On the one hand, Davenport's technique in [4] currently admits no generalization to higher dimensions. Indeed, the proof of a three-dimensional analog of Theorem 2 along the line of Davenport's argument would require the falsity of the celebrated conjecture of Littlewood on diophantine approximation. On the other hand, although the van der Corput sequence enables one to give the best possible upper bound for the local discrepancy (1.1), it is insufficient on its own even to give a proof of Theorem 2, let alone any higher dimensional analog. In fact, we show in Theorem 3 that the van der Corput sequence gives an estimate of higher order of magnitude than for the integral in Theorem 2.

In [16], Roth used the van der Corput sequence as generalized by Halton [8] and Hammersley [9]. He introduced a powerful probabilistic argument to obtain higher dimensional analogs of Theorem 2, at the expense of the explicitness of the point sets $\mathcal{B}_N$ involved. In fact, currently in the literature there is no explicit construction of point sets $\mathcal{B}_N$ that satisfy any higher dimensional analog of Theorem 2.

The starting point of our investigation in the present paper is to examine many of the ideas in various proofs of Theorem 2. While Davenport's original proof gives an explicit point set $\mathcal{B}_N$, all subsequent proofs, by Roth [14–16], Chen [2], Dobrovol'skiĭ [5], Skriganov [18, 19], and Beck and Chen [1], are probabilistic in nature and

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do not give any explicit point sets. We indicate briefly how the idea of reflection introduced by Davenport in [4] can be combined with the van der Corput point sets to give an explicit proof of Theorem 2.

We also discuss a new group-theoretic approach to the problem, arising from the observation that some van der Corput point sets have a very nice group structure. It is then natural to use the group characters, which turn out to be Walsh functions. We shed new light on why the van der Corput sequence on its own is unsuitable for proving Theorem 2. We also combine this with Davenport's reflection principle to give another explicit proof of Theorem 2. Although this approach does not give any new results in dimension 2, its major benefit is that, with extra ideas, it can be generalized to higher dimensions, which will be discussed in a forthcoming paper (see [3]).

The present paper is organized as follows. In Sec. 2, we highlight the idea on reflection that is crucial to Davenport's proof of Theorem 2. In Sec. 3, we introduce the van der Corput point sets, state a theorem to show that they are insufficient to give a proof of Theorem 2, and indicate briefly how they can be modified to give proofs of Theorem 2. In Sec. 4, we introduce the Walsh functions and combine them with the van der Corput point sets. In Sec. 5, we study the idea on reflection more carefully. We then combine our ideas in Sec. 6 to sketch a new explicit proof of Theorem 2. In view of our forthcoming paper [3], the discussion in Sec. 6 is restricted to a very brief sketch, because the details are technically complicated. Finally, we indicate the limitations of the van der Corput point sets in Sec. 7.

For convenience, we denote by $\mathbb{N}$ the set of all positive integers, by $\mathbb{N}_0$ the set of all nonnegative integers, by $\mathbb{Q}$ the set of all rational numbers, and by $\mathbb{R}$ the set of all real numbers. For any $\beta \in \mathbb{R}$, we let $\{\beta\}$ denote the fractional part of β, $\phi(\beta) = \{\beta\} - 1/2$ if $\beta \notin \mathbb{Z}$, $\phi(\beta) = 0$ if $\beta \in \mathbb{Z}$, and $e(\beta) = e^{2\pi i\beta}$. If $\mathcal{S}$ is a set, then $\chi_{\mathcal{S}}$ denotes its characteristic function. If $\mathcal{S}$ is a finite set, then $\#(\mathcal{S})$ denotes the number of elements in $\mathcal{S}$.

Remark. Larcher [10] independently pursued many ideas from the present paper and gave an explicit construction satisfying a three-dimensional analog of Theorem 2. However, difficulties with binomial coefficients prevent a generalization to dimension 4 and higher dimensions.

2. The ideas of Davenport

In this section, we give a brief sketch of Davenport's ideas in [4].

Consider a lattice Λ on the plane generated by the two vectors $(1,0)$ and $(\theta,1)$, where θ is an irrational number. Assume that M is a positive integer. We are interested in the set $\mathcal{Q}$ that contains precisely M points of Λ that fall into the rectangle $[0,1) \times [0,M)$. Clearly,

$$\mathcal{Q} = \{(\{\theta n\}, n) : 0 \leq n \leq M-1\}.$$

Next, we consider a rectangle of the form

$$R(y_1,y_2) = [0,y_1) \times [0,y_2) \subseteq [0,1) \times [0,M),$$

where y_1 is arbitrary and y_2 is an integer. It is not difficult to show that

$$\#(\mathcal{Q} \cap R(y_1,y_2)) - y_1y_2 = \sum_{n=0}^{y_2-1} (\phi(\theta n - y_1) - \phi(\theta n))$$

for all but finitely many values of y_1 in the interval $[0,1)$. This has the Fourier expansion

$$\sum_{m\neq 0} \left(\frac{1-e(-my_1)}{2\pi i m}\right)\left(\sum_{n=0}^{y_2-1} e(\theta n m)\right). \tag{2.1}$$

Ideally, we would like to square expression (2.1) and integrate with respect to y_1 over the interval $[0,1)$. Unfortunately, the term 1 in the numerator $1-e(-my_1)$ proves to be a nuisance. In order to overcome this difficulty, we consider the lattice Λ' on the plane generated by the two vectors $(1,0)$ and $(-\theta,1)$. Then

$$\mathcal{Q}' = \{(\{-\theta n\}, n) : 0 \leq n \leq M-1\}$$

is the set containing precisely M points of Λ' that fall into the rectangle $[0,1)\times[0,M)$. If we now consider the $2M$ points of $\mathcal{Q}\cup\mathcal{Q}'$ that fall into this rectangle, then it is not difficult to show that

$$\#((\mathcal{Q}\cup\mathcal{Q}')\cap R(y_1,y_2))-2y_1y_2=\sum_{n=0}^{y_2-1}(\phi(\theta n-y_1)-\phi(\theta n+y_1))$$

for all but finitely many values of y_1 in the interval $[0,1)$, and we obtain the Fourier expansion

$$\sum_{m\neq 0}\left(\frac{e(my_1)-e(-my_1)}{2\pi im}\right)\left(\sum_{n=0}^{y_2-1}e(\theta nm)\right).$$

It follows from Parseval's theorem that

$$\int_0^1|\#((\mathcal{Q}\cup\mathcal{Q}')\cap R(y_1,y_2))-2y_1y_2|^2\,dy_1\ll\sum_{m=1}^{\infty}\frac{1}{m^2}\left|\sum_{n=0}^{y_2-1}e(\theta nm)\right|^2. \tag{2.2}$$

Now, we assume that the irrational number θ has a continued fraction expansion with bounded partial quotients. It can be shown that the sum on the right-hand side of (2.2) is bounded from above by a constant multiple of $\log(2M)$. On the other hand, if we lift the restriction that y_2 is an integer, we obtain an error of $O(1)$, where the implicit constant depends at most on θ. It follows trivially that

$$\int_0^M\int_0^1|\#((\mathcal{Q}\cup\mathcal{Q}')\cap R(y_1,y_2))-2y_1y_2|^2\,dy_1dy_2\ll M\log(2M),$$

where the implicit constant depends at most on θ. Rescaling in the y_2 direction by a factor $1/M$, we see that the set

$$\mathcal{P}=\{(\{\pm\theta n\},n/M):0\le n\le M-1\}$$

of $N=2M$ points in $[0,1)^2$ satisfies the assumptions of Davenport's theorem.

Note that the lattices Λ and Λ' are symmetric through the vertical line $2y_1=1$.

3. Van der Corput sets

Now, let $s\in\mathbb{N}$. In U^2, we consider the van der Corput set of 2^s points given by

$$\mathcal{P}(s)=\left\{\left(\sum_{i=1}^{s}a_i2^{-i},\sum_{i=1}^{s}a_{s+1-i}2^{-i}\right):a_1,a_2\dots,a_s\in\{0,1\}\right\}.$$

The following result shows that this set is not sufficiently good to give the minimal order of the L_2-discrepancy.

Theorem 3. *For every $s\in\mathbb{N}$, we have*

$$\int_{U^2}|\#(\mathcal{P}(s)\cap B_Y)-2^sy_1y_2|^2\,dY=2^{-6}s^2+O(s).$$

We note here that a lower bound without the specific constant 2^{-6} was given by Matoušek (see Section 2.2 of [11]). For this reason, we only give a very brief sketch of the proof in Sec. 7 to indicate how this constant arises. Here, we discuss briefly how this set can be modified to give a proof of Theorem 2.

Definition. *Assume that $s\in\mathbb{N}_0$. By an s-box, we mean a rectangle of the form*

$$[m_12^{-i_1},(m_1+1)2^{-i_1})\times[m_22^{-i_2},(m_2+1)2^{-i_2})\subseteq U^2,$$

where $m_1,m_2,i_1,i_2\in\mathbb{N}_0$ satisfy the condition $i_1+i_2=s$, so that the rectangle has area 2^{-s}.

The following simple observation stems immediately from the definition of van der Corput sets and s-boxes.

Lemma 3A. *Assume that $s \in \mathbb{N}_0$. Then every s-box contains precisely one point of the van der Corput set $\mathcal{P}(s)$.*

It is convenient to rescale in the y_2 direction. Accordingly, we consider the set

$$\mathcal{Q}(s) = \left\{ \left(\sum_{i=1}^{s} a_i 2^{-i}, \sum_{i=1}^{s} a_{s+1-i} 2^{s-i} \right) : a_1, a_2 \ldots, a_s \in \{0,1\} \right\} \subset [0,1) \times [0, 2^s).$$

Assume that $y_1 \in [0,1)$ is a fixed integer multiple of 2^{-s}. Then it can be shown that, for almost all real numbers $y_2 \in [0, 2^s)$, we have

$$\#(\mathcal{Q}(s) \cap ([0,y_1) \times [0,y_2))) - y_1 y_2 = \sum_{i \in \mathcal{I}} \left(c_i + \phi\left(\frac{z_i - y_2}{L_i} \right) \right),$$

where $\mathcal{I}$ is a finite set of indices dependent only on y_1 and where the integer L_i is a divisor of 2^s for every $i \in \mathcal{I}$. Therefore, to study L_2-discrepancy, we need to consider sums of the form

$$\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} \left(c_i + \phi\left(\frac{z_i - y_2}{L_i} \right) \right) \left(c_j + \phi\left(\frac{z_j - y_2}{L_j} \right) \right). \tag{3.1}$$

Integrating this sum with respect to y_2 over the interval $[0, 2^s)$, we see that each summand in (3.1) gives rise to the integral

$$\int_0^{2^s} \left(c_i + \phi\left(\frac{z_i - y_2}{L_i} \right) \right) \left(c_j + \phi\left(\frac{z_j - y_2}{L_j} \right) \right) dy_2 = 2^s \left(c_i c_j + O\left(\frac{(L_i, L_j)^2}{L_i L_j} \right) \right),$$

where, for every $i, j \in \mathcal{I}$, the term (L_i, L_j) denotes the greatest common divisor of L_i and L_j. Unfortunately, the term $c_i c_j$ is a serious handicap to any further progress.

Remark. The constants c_i can be calculated explicitly. A careful analysis using the above leads to a proof of Theorem 3.

The idea of Roth in [16] is to introduce a translation variable t in the y_2 direction, and consider point sets of the form

$$\mathcal{Q}(s;t) = \{(x_1, x_2 + t) : (x_1, x_2) \in \mathcal{Q}(s)\},$$

where the addition $x_2 + t$ is modulo 2^s. Then

$$\#(\mathcal{Q}(s;t) \cap ([0,y_1) \times [0,y_2)) - y_1 y_2 = \sum_{i \in \mathcal{I}} \left(\phi\left(\frac{z_i - y_2 + t}{L_i} \right) - \phi\left(\frac{z_i' - y_2 + t}{L_i} \right) \right).$$

Squaring the sum on the right-hand side and integrating over t, we see that every summand gives rise to four integrals of the form

$$\int_0^{2^s} \phi\left(\frac{z_i - y_2 + t}{L_i} \right) \phi\left(\frac{z_j - y_2 + t}{L_j} \right) dt = O\left(2^s \frac{(L_i, L_j)^2}{L_i L_j} \right).$$

Alternatively, we can follow the ideas of Proinov [12] and consider the reflected set

$$\mathcal{Q}'(s) = \{(x_1, 2^s - x_2) : (x_1, x_2) \in \mathcal{Q}(s)\},$$

where the subtraction $2^s - x_2$ is modulo 2^s. Then it can be shown that, for almost all real numbers $y_2 \in [0, 2^s)$, we have

$$\#(\mathcal{Q}'(s) \cap ([0,y_1) \times [0,y_2)) - y_1 y_2 = \sum_{i \in \mathcal{I}} \left(-c_i + \phi\left(\frac{z_i' - y_2}{L_i} \right) \right),$$

so that the combined local discrepancy is given by

$$\#((\mathcal{Q}(s) \cup \mathcal{Q}'(s)) \cap ([0,y_1) \times [0,y_2)) - 2 y_1 y_2 = \sum_{i \in \mathcal{I}} \left(\phi\left(\frac{z_i - y_2}{L_i} \right) + \phi\left(\frac{z_i' - y_2}{L_i} \right) \right).$$

Squaring the sum on the right-hand side and integrating over y_2, we see that every summand gives rise to four integrals of the form

$$\int_0^{2^s} \phi\left(\frac{z_i - y_2}{L_i}\right) \phi\left(\frac{z_j - y_2}{L_j}\right) dt = O\left(2^s \frac{(L_i, L_j)^2}{L_i L_j}\right).$$

Therefore, we can conclude that the set $\mathcal{Q}(s) \cup \mathcal{Q}'(s)$ of 2^{s+1} points satisfies the equation

$$\int_0^{2^s} \int_0^1 |\#((\mathcal{Q}(s) \cup \mathcal{Q}'(s)) \cap ([0, y_1) \times [0, y_2))) - 2y_1 y_2|^2 \, dy_1 dy_2 = O(2^s s).$$

This leads to explicit constructions satisfying the conclusion of Theorem 2. We omit the details here.

4. The unit interval and Walsh functions

For any $s \in \mathbb{N}_0$, let $\mathbb{Q}(2^s) = \{m2^{-s} : m = 0, 1, \ldots, 2^s - 1\}$, and let

$$\mathbb{Q}(2^\infty) = \bigcup_{s=0}^{\infty} \mathbb{Q}(2^s)$$

denote the binary rational numbers.

We observe that every x in $[0, 1)$ can be represented in the form

$$x = \sum_{i=1}^{\infty} \eta_i(x) 2^{-i}, \tag{4.1}$$

where $\eta_i(x) \in \{0, 1\}$ for every $i \in \mathbb{N}$. This representation is unique if we agree that the series in (4.1) is finite for every $x \in \mathbb{Q}(2^\infty)$. For any two elements $x, y \in \mathbb{Q}(2^\infty)$, we can write

$$x \oplus y \in \mathbb{Q}(2^\infty) \tag{4.2}$$

by setting

$$\eta_i(x \oplus y) = \eta_i(x) + \eta_i(y) \pmod 2$$

for every $i \in \mathbb{N}$. It is easy to see that, with respect to operation (4.2), each set $\mathbb{Q}(2^s)$ forms a finite Abelian group, while the set $\mathbb{Q}(2^\infty)$ forms an infinite Abelian group. It is well known that the characters of these groups are Walsh functions.

Each ℓ in $\mathbb{N}_0$ can uniquely be written in the form

$$\ell = \sum_{i=1}^{\infty} \lambda_i(\ell) 2^{i-1}, \tag{4.3}$$

where $\lambda_i(\ell) \in \{0, 1\}$ for every $i \in \mathbb{N}$. For every real number $x \in [0, 1)$ of the form (4.1), we consider the Walsh function

$$w_\ell(x) = (-1)^{\sum_{i=1}^{\infty} \lambda_i(\ell)\eta_i(x)}. \tag{4.4}$$

A detailed study of these functions can be found in [7] and [17].

Since (4.3) is essentially a finite sum, the function $w_\ell(x)$ is well defined and takes the values ± 1. We have $w_0(x) = 1$ for every $x \in [0, 1)$ and

$$\int_0^1 w_\ell(y) \, dy = 0$$

for every $\ell \in \mathbb{N}$. In other words, with one exception $w_0(x)$, all Walsh functions have zero mean over the unit interval. Of particular importance is the fact that the Walsh functions form an orthonormal basis of $L_2([0, 1))$.

It is easy to see that, for every $s \in \mathbb{N}$, the van der Corput set $\mathcal{P}(s)$ of 2^s points in U^2 forms a group under operation (4.2) for each coordinate. On the other hand, we can combine Lemma 3A with properties of Walsh functions to establish a number of results. In the following two lemmas, we use the notation $X = (x_1, x_2)$.

Lemma 4A. *Assume that $\mathcal{P}(s)$ is the van der Corput set of 2^s points in U^2. Then, for all $i, q \in \mathbb{N}_0$ satisfying $i < s$ and $0 \le q < 2^i$, we have*

$$\sum_{X \in \mathcal{P}(s)} w_{2^i \oplus q}(x_1) = 0.$$

Lemma 4B. *Assume that $\mathcal{P}(s)$ is the van der Corput set of 2^s points in U^2. Then, for all $i_1, i_2 \in \mathbb{N}_0$, the following assertions are valid:*

(i) *Let $i_1 + i_2 < s - 1$. Then, for all $q_1, q_2 \in \mathbb{N}_0$ satisfying $0 \le q_1 < 2^{i_1}$ and $0 \le q_2 < 2^{i_2}$, we have*

$$\sum_{X \in \mathcal{P}(s)} w_{2^{i_1} \oplus q_1}(x_1) w_{2^{i_2} \oplus q_2}(x_2) = 0.$$

(ii) *We have*

$$\sum_{X \in \mathcal{P}(s)} w_{2^{i_1}}(x_1) w_{2^{i_2}}(x_2) \chi_{[0,2^{-i_1})}(x_1) \chi_{[0,2^{-i_2})}(x_2) = \begin{cases} 0 & \text{if } i_1 + i_2 \le s - 2, \\ 1 & \text{if } i_1 + i_2 \ge s \end{cases}$$

and

$$\left| \sum_{X \in \mathcal{P}(s)} w_{2^{i_1}}(x_1) w_{2^{i_2}}(x_2) \chi_{[0,2^{-i_1})}(x_1) \chi_{[0,2^{-i_2})}(x_2) \right| \le 2 \text{ if } i_1 + i_2 = s - 1.$$

(iii) *We have*

$$\left| \sum_{X \in \mathcal{P}(s)} w_{2^{i_1}}(x_1) \chi_{[0,2^{-i_1})}(x_1) w_{2^{i_2}}(x_2) \right| \le \max\{1, 2^{s-i_1}\}.$$

5. The Davenport reflection

Let $\mathcal{P}(s)$ be the van der Corput set of 2^s points in U^2. Consider the mapping

$$\Theta : U^2 \to [0,1]^2 : (x_1, x_2) \mapsto (|2x_1 - 1|, |2x_2 - 1|). \tag{5.1}$$

We study the point set $\mathcal{P}^*(s) = \Theta(\mathcal{P}(s))$, where the points are counted with multiplicity, so that $\mathcal{P}^*(s)$ is a distribution of 2^s points in $[0,1]^2$. Note that except for the point $\Theta(0,0) = (1,1)$, all other points of $\mathcal{P}^*(s)$ lie in $[0,1)^2$.

Suppose $Y = (y_1, y_2) \in [0, 1/2)^2$. Then the two rectangles

$$B_Y^{\mathrm{mod}} = (y_1, 1 - y_1) \times (y_2, 1 - y_2)$$

and

$$B_{\Theta(Y)} = [0, 1 - 2y_1) \times [0, 1 - 2y_2)$$

have the same area $(1 - 2y_1)(1 - 2y_2)$. Furthermore, for every point $X \in \mathcal{P}(s)$ we have

$$X \in B_Y^{\mathrm{mod}} \qquad \text{if and only if} \qquad \Theta(X) \in B_{\Theta(Y)}.$$

Hence the quantities given by

$$\mathcal{L}^{\mathrm{mod}}[\mathcal{P}(s); Y] = \#(\mathcal{P}(s) \cap B_Y^{\mathrm{mod}}) - 2^s(1 - 2y_1)(1 - 2y_2)$$

and

$$\mathcal{L}[\mathcal{P}^*(s); \Theta(Y)] = \#(\mathcal{P}^*(s) \cap B_{\Theta(Y)}) - 2^s(1 - 2y_1)(1 - 2y_2)$$

are equal, and with the help of the substitution $V = \Theta(Y)$ it follows that

$$\int_{U^2} |\mathcal{L}[\mathcal{P}^*(s); V]|^2 \, dV = 4 \int_{[0,1/2)^2} |\mathcal{L}^{\mathrm{mod}}[\mathcal{P}(s); Y]|^2 \, dY. \tag{5.2}$$

Theorem 2 can easily be deduced from the following result.

Lemma 5A. *Assume that $\mathcal{P}(s)$ is the van der Corput set of 2^s points in U^2. Then*

$$\int_{[0,1/2)^2} |\mathcal{L}^{\mathrm{mod}}[\mathcal{P}(s); Y]|^2 \, dY \le 300s.$$

6. A sketch of the proof of Lemma 5A

In what follows, $\mathcal{P} = \mathcal{P}(s)$ denotes the van der Corput set of 2^s points in U^2. Since

$$\#(\mathcal{P} \cap B_Y^{\mathrm{mod}}) = \sum_{X \in \mathcal{P}} \chi_{B_Y^{\mathrm{mod}}}(X) = \sum_{X \in \mathcal{P}} \chi_{(y_1, 1-y_1)}(x_1) \chi_{(y_2, 1-y_2)}(x_2),$$

our first step is, naturally, to describe the characteristic function $\chi_{(y,1-y)}(x)$ in terms of Walsh functions.

Assume that $y \in [0,1)$ is fixed. Since the Walsh functions form an orthonormal basis of $L_2([0,1))$, it follows that, for every $x \in [0,1)$, we have

$$\chi_{[0,y)}(x) \simeq \sum_{\ell=0}^{\infty} \tilde{\chi}_\ell(y) w_\ell(x), \tag{6.1}$$

where the symbol $\simeq$ denotes that series (6.1) converges in the L_2-norm and where, for every $\ell \in \mathbb{N}_0$, we have

$$\tilde{\chi}_\ell(y) = \int_0^1 \chi_{[0,y)}(x) w_\ell(x)\, dx = \int_0^y w_\ell(x)\, dx. \tag{6.2}$$

In particular, we have

$$\tilde{\chi}_0(y) = \int_0^y w_0(x)\, dx = y. \tag{6.3}$$

On the other hand, there exists a unique $m \in \mathbb{N}_0$ such that $y \in [m2^{-s}, (m+1)2^{-s})$. Consider the function

$$\chi_{[0,y)}^{(s)}(x) = \begin{cases} 1 & \text{if } 0 \le x < m2^{-s}, \\ 2^s y - m & \text{if } m2^{-s} \le x < (m+1)2^{-s}, \\ 0 & \text{if } (m+1)2^{-s} \le x < 1, \end{cases}$$

where

$$2^s y - m = 2^s \int_{m2^{-s}}^{(m+1)2^{-s}} \chi_{[0,y)}(x)\, dx$$

represents the average value of $\chi_{[0,y)}(x)$ in the interval $[m2^{-s}, (m+1)2^{-s})$. It is well known that

$$\chi_{[0,y)}^{(s)}(x) = \sum_{\ell=0}^{2^s - 1} \tilde{\chi}_\ell(y) w_\ell(x). \tag{6.4}$$

For details, see Section 2.8 in [7]. It is easy to see that $|\chi_{[0,y)}(x) - \chi_{[0,y)}^{(s)}(x)|$ is equal to 0 whenever $x \notin [m2^{-s}, (m+1)2^{-s})$, and is at most 1 always.

The evaluation of integral (6.2) was studied by Fine [6]. For every $\ell \in \mathbb{N}$, we have the identity

$$\tilde{\chi}_\ell(y) = \frac{1}{4} 2^{-\nu(\ell)} \left(w_{\ell \oplus 2^{\nu(\ell)}}(y) - \sum_{j=1}^{\infty} 2^{-j} w_{\ell \oplus 2^{\nu(\ell)+j}}(y) \right), \tag{6.5}$$

where $\nu(\ell) \in \mathbb{N}_0$ denotes the unique integer satisfying $2^{\nu(\ell)} \le \ell < 2^{\nu(\ell)+1}$. For the special case where $\ell = 2^i$ with $i \in \mathbb{N}_0$, we have $\ell \oplus 2^{\nu(\ell)} = 0$, and thus the function $\tilde{\chi}_\ell(y)$ has nonzero mean value over the interval $[0,1)$. Therefore, we introduce the quantity

$$\delta_\ell = \begin{cases} 1 & \text{if } \ell = 2^i \text{ for some } i \in \mathbb{N}_0, \\ 0 & \text{if } \ell \ne 2^i \text{ for any } i \in \mathbb{N}_0, \end{cases}$$

and study the function

$$\Omega_\ell(y) = w_{\ell \oplus 2^{\nu(\ell)}}(y) - \delta_\ell - \sum_{j=1}^{\infty} 2^{-j} w_{\ell \oplus 2^{\nu(\ell)+j}}(y). \tag{6.6}$$

The need to handle the extra term δ_ℓ is the reason for introducing reflection mapping (5.1). Suppose $y \in [0,1/2)$. Then, for almost all $x \in [0,1)$, we have $\chi_{(y,1-y)}(x) = \chi_{[0,1-y)}(x) - \chi_{[0,y)}(x)$. Now let

$$\chi^{(s)}_{(y,1-y)}(x) = \chi^{(s)}_{[0,1-y)}(x) - \chi^{(s)}_{[0,y)}(x).$$

Then almost always, we have

$$\chi^{(s)}_{(y,1-y)}(x) = 1 - 2y + \sum_{\ell=1}^{2^s-1} (\tilde{\chi}_\ell(1-y) - \tilde{\chi}_\ell(y)) w_\ell(x) = 1 - 2y + \frac{1}{4} \sum_{\ell=1}^{2^s-1} 2^{-\nu(\ell)} (\Omega_\ell(1-y) - \Omega_\ell(y)) w_\ell(x).$$

On the other hand, it is easy to see that if we write

$$\chi^{(s)}_{B^{\mathrm{mod}}_Y}(X) = \chi^{(s)}_{(y_1,1-y_1)}(x_1) \chi^{(s)}_{(y_2,1-y_2)}(x_2),$$

then $|\chi_{B^{\mathrm{mod}}_Y}(X) - \chi^{(s)}_{B^{\mathrm{mod}}_Y}(X)|$ is equal to 0 if X does not belong to four s-boxes, and is at most 2 always. It follows that if we write

$$\mathcal{M}^{\mathrm{mod}}[\mathcal{P};Y] = \sum_{X \in \mathcal{P}} \chi^{(s)}_{B^{\mathrm{mod}}_Y}(X) - 2^s(1-2y_1)(1-2y_2), \tag{6.7}$$

then the inequality $|\mathcal{L}^{\mathrm{mod}}[\mathcal{P};Y]| \le |\mathcal{M}^{\mathrm{mod}}[\mathcal{P};Y]| + 8$ is fulfilled for almost all $Y \in [0,1/2)^2$, so that

$$\int_{[0,1/2)^2} |\mathcal{L}^{\mathrm{mod}}[\mathcal{P};Y]|^2 \, dY \le 2 \int_{[0,1/2)^2} |\mathcal{M}^{\mathrm{mod}}[\mathcal{P};Y]|^2 \, dY + 128.$$

Now the proof of Lemma 5A is reduced to finding a suitable upper bound for the integral

$$\int_{[0,1/2)^2} |\mathcal{M}^{\mathrm{mod}}[\mathcal{P};Y]|^2 \, dY.$$

We omit the long and technical calculations here.

7. A sketch of the proof of Theorem 3

It follows from (6.3)–(6.6) that, for every $y \in [0,1)$, we have

$$\chi^{(s)}_{[0,y)}(x) = y + \frac{1}{4} \sum_{i=0}^{s-1} 2^{-i} w_{2^i}(x) + \frac{1}{4} \sum_{\ell=1}^{2^s-1} 2^{-\nu(\ell)} \Omega_\ell(y) w_\ell(x).$$

Using this relation, we can show, similarly to (6.7), that the local discrepancy $\mathcal{L}[\mathcal{P};Y]$ can be approximated by a function of the form

$$\begin{aligned}\mathcal{M}[\mathcal{P};Y] = \frac{1}{4} y_2 \sum_{i_1=0}^{s-1} 2^{-i_1} \sum_{X \in \mathcal{P}} w_{2^{i_1}}(x_1) + \frac{1}{4} y_1 \sum_{i_2=0}^{s-1} 2^{-i_2} \sum_{X \in \mathcal{P}} w_{2^{i_2}}(x_2) \\ + \frac{1}{16} \sum_{i_1=0}^{s-1} \sum_{i_2=0}^{s-1} 2^{-i_1-i_2} \sum_{X \in \mathcal{P}} w_{2^{i_1}}(x_1) w_{2^{i_2}}(x_2) + \mathcal{Z}[\mathcal{P};Y],\end{aligned}$$

where the function $\mathcal{Z}[\mathcal{P};Y]$ has zero mean over $Y \in U^2$ and satisfies the condition

$$\int_{U^2} |\mathcal{Z}[\mathcal{P};Y]|^2 \, dY = O(s).$$

It can be shown that if $i_1 \le s-1$ and $i_2 \le s-1$, then

$$\sum_{X \in \mathcal{P}} w_{2^{i_1}}(x_1) w_{2^{i_2}}(x_2) = \begin{cases} 2^s & \text{if } i_1 + i_2 = s-1, \\ 0 & \text{otherwise.} \end{cases}$$

Using this relation and Lemma 4A, we see that

$$\mathcal{M}[\mathcal{P};Y] = \frac{1}{16} \sum_{\substack{i_1=0 \\ i_1+i_2=s-1}}^{s-1} \sum_{i_2=0}^{s-1} 2^{s-i_1-i_2} + \mathcal{Z}[\mathcal{P};Y] = 2^{-3}s + \mathcal{Z}[\mathcal{P};Y].$$

This yields

$$\int_{U^2} |\mathcal{M}[\mathcal{P};Y]|^2 \, dY = 2^{-6}s^2 + O(s).$$

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