RESULTS AND PROBLEMS OLD AND NEW IN DISCREPANCY THEORY

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Dedicated to the memory of my grandson Alexander

1. Introduction

The subject of discrepancy theory, or irregularities of point distribution, began with the conjecture of van der Corput [29, 30] in 1935 and the pioneering results of van Aardenne-Ehrenfest [1, 2] in 1945 and 1949, and took on a geometric flavour through the groundbreaking early work of Roth [43] in 1954. Today, many of the problems are formulated in the following way.

Let U be a bounded region in the k-dimensional euclidean space $\mathbb{R}^k$, where $k \geqslant 2$, endowed with a measure μ, usually the Lebesgue measure, and let $\mathscr{P}$ be a set of N points in U. The irregularity of the distribution of the point set $\mathscr{P}$ is usually described in terms of an infinite collection $\mathcal{B}$ of measurable sets in U. For any such measurable set B in $\mathcal{B}$, we consider the discrepancy function

$$D[\mathscr{P};B] = |\mathscr{P} \cap B| - N\mu(B).$$

Often the collection $\mathcal{B}$ is endowed with an integral geometric measure $\mathrm{d}B$. Then for any real number q satisfying $0 < q < \infty$, we can consider the L_q-discrepancy

$$\|D_{\mathcal{B}}(\mathscr{P})\|_q = \left(\int_{\mathcal{B}} |D[\mathscr{P};B]|^q \,\mathrm{d}B\right)^{1/q}.$$

Here the values $q = 1$ and $q = 2$ are often of particular interest. We also consider the L_∞-discrepancy, or extreme discrepancy,

$$\|D_{\mathcal{B}}(\mathscr{P})\|_\infty = \sup_{B \in \mathcal{B}} |D[\mathscr{P};B]|.$$

Our goal is then to find lower and upper bounds for the quantities

$$\inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_q \quad \text{and} \quad \inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_\infty,$$

where each infimum is taken over all points sets $\mathscr{P}$ of N points in U.

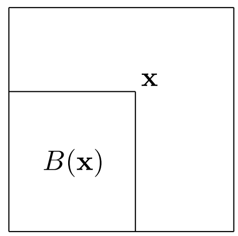
Notation. For any function f and positive function g, we write $f \ll g$ to denote that there exists a positive constant C such that $|f| \leqslant Cg$. In particular, if f is a positive function, then we also write $f \gg g$ to indicate that $g \ll f$, and write $f \asymp g$ to indicate that both $f \ll g$ and $f \gg g$ hold. The signs $\ll$, $\gg$ and $\asymp$ may contain subscripts, denoting that any implicit constants that arise may depend on these parameters. For any finite set S, we write $|S|$ to denote the cardinality of S.

2. The Classical Problem

The classical problem in discrepancy theory was formulated by Roth [43] in 1954. Here $U = [0,1]^k$, the unit cube in $\mathbb{R}^k$, where $k \geqslant 2$, and the irregularity of a point set $\mathscr{P}$ in U is described in terms of the infinite collection

$$\mathcal{B} = \{B(\mathbf{x}) = [0,x_1) \times \ldots \times [0,x_k) : \mathbf{x} \in [0,1]^k\}$$

of aligned rectangular boxes in the unit cube anchored at the origin.



The integral geometric measure in $\mathcal{B}$ is given by the usual Lebesgue volume measure $\mathrm{d}B = \mathrm{d}\mathbf{x}$.

The L_q-discrepancy in this problem is well understood for every real number q satisfying $1 < q < \infty$, and we have the estimates

$$(\log N)^{(k-1)/2} \ll_{k,q} \inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_q \ll_{k,q} (\log N)^{(k-1)/2}. \tag{1}$$

Here the lower bound is due to Schmidt [51] in 1977, following the earlier work of Roth [43] in 1954 on the special case $q = 2$ using an orthogonal function technique. The upper bound is due to Chen [19, 20], following the earlier work of Davenport [31] in 1956 on the special case $k = q = 2$ and the big breakthrough of Roth [45] in 1980 on the special case $q = 2$.

We make here a few comments concerning the special case $q = 2$.

The proof of the lower bound is given in Roth [43] for the case $k = 2$ only, although generalization to arbitrary dimensions $k \geqslant 2$ presents no extra difficulties. In fact, the ideas are much more clearly presented in Schmidt [51]. A complete proof of these results of Roth and Schmidt in arbitrary dimensions can be found in the monograph of Beck and Chen [9, Section 2.1]. However, a simple description of the ideas along these lines for the case $k = 2$ can be found in the survey of Chen and Travaglini [26, Section 1]. The idea is that sets where the expectation is a small fraction between 0 and 1 can be found in abundance, and they give rise to what we call *trivial discrepancies*. We need to combine these and not allow them to cancel each other. The tool is given by Roth's auxiliary function, of the form

$$F(\mathbf{x}) = \sum_{\mathbf{r}} f_{\mathbf{r}}(\mathbf{x}),$$

a sum of orthogonal functions over a suitable collection of vectors $\mathbf{r}$, where each $f_{\mathbf{r}}(\mathbf{x})$ is either a Rademacher type function with values ± 1 or zero. Writing $D(\mathbf{x})$ for $D[\mathscr{P}; B(\mathbf{x})]$, the Cauchy–Schwarz inequality then gives

$$\left| \int_U D(\mathbf{x}) F(\mathbf{x}) \, \mathrm{d}\mathbf{x} \right| \leqslant \|D\|_2 \|F\|_2. \tag{2}$$

A lower bound for $\|D\|_2$ will result from a lower bound for the left hand side of (2) and an upper bound for $\|F\|_2$.

The proof of the upper bound in Roth [45] is probabilistic, with no explicitly given point sets, as are subsequent proofs by Chen [20] in 1983 and Skriganov [53] in 1994. The first proof of the upper bound with explicitly given point sets can be found in Chen and Skriganov [24] in 2002. A different proof is given by Dick and Pillichshammer [32] in 2014. For some comments on the differences between these two explicit proofs, see also the paper of Dick and Pillichshammer [33].

On the other hand, for the case $q = 1$, we have the estimates

$$(\log N)^{1/2} \ll_k \inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_1 \ll_k (\log N)^{(k-1)/2}.$$

Here the upper bound is a simple consequence of the upper bound in (1), while the lower bound is due to Halász [34] in 1981, using a variant of Roth's lower bound technique that only works when $k = 2$. Indeed, Halász uses the auxiliary function

$$H(\mathbf{x}) = \prod_{\mathbf{r}} \left(1 + \mathrm{i}n^{-1/2} f_{\mathbf{r}}(\mathbf{x})\right) - 1,$$

where $\log N \ll n \ll \log N$. Then $H(\mathbf{x}) \ll 1$, and so

$$\left| \int_U D(\mathbf{x}) H(\mathbf{x}) \, \mathrm{d}\mathbf{x} \right| \ll \|D\|_1. \tag{3}$$

A lower bound for $\|D\|_1$ will result from a lower bound for the left hand side of (3).

Thus the problem of the L_q-discrepancy in this classical setting is completely solved for all finite $q > 1$ and for the case $(k, q) = (2, 1)$.

Clearly the upper bound in (1) remains valid for every natural number $k \geqslant 2$ and every finite positive real number q.

Open Problem 1. *In the classical discrepancy problem, is it true that for every natural number $k \geqslant 2$ and every finite positive real number q, the estimate*

$$\inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_q \gg_{k,q} (\log N)^{(k-1)/2} \tag{4}$$

holds?

It is interesting to observe that for every real number q satisfying $0 < q < 1$, the currently known best lower bound is precisely zero.

Much less is known for the L_∞-discrepancy. We have the estimates

$$(\log N)^{(k-1)/2} \ll_k \inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_\infty \ll_k (\log N)^{k-1}. \tag{5}$$

Here the lower bound is a simple consequence of the lower bound in (1), while the upper bound is due to Halton [35] in 1960. The lower bound has been improved in the intervening years, and we have the estimate

$$\inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_\infty \gg \log N \tag{6}$$

in the special case $k = 2$, due first to Schmidt [49] in 1972, using a combinatorial argument, with an alternative proof given by Halász [34] in 1981, using Roth's technique with yet another auxiliary function, as well as the estimate

$$\inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_\infty \gg_k (\log N)^{(k-1)/2+\delta(k)} \tag{7}$$

for some $\delta(k) \in (0, 1/2)$, due to Bilyk, Lacey and Vagharshakyan [15] in 2008.

There remains a rather big gap between the lower and upper bounds when $k > 2$.

Open Problem 2 (Great Open Problem)**.** *In the classical discrepancy problem, for every natural number $k \geqslant 3$, find the correct order of magnitude of*

$$\inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_\infty.$$

At the very least, try to prove or disprove the following conjectures:

(i) (Old Conjecture) *For every natural number $k \geqslant 3$, the estimate*

$$\inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_\infty \gg_k (\log N)^{k-1}$$

holds, so that Halton's upper bound in (5) *is sharp.*

(ii) (New Conjecture) *For every natural number $k \geqslant 3$, the estimate*

$$\inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_\infty \gg_k (\log N)^{k/2} \tag{8}$$

holds, corresponding to the estimate in (7) *with $\delta(k) = 1/2$.*

We comment that both estimates (4) and (8) hold *on average* over digit shifts, as shown by Skriganov [54] in 2016. Digit shifts, since its introduction to discrepancy theory by Chen [20] in 1983, have always been used to study upper bound questions. This recent work of Skriganov is the first instance that they have been used in lower bound considerations.

Before we make our concluding remarks, we mention a very interesting piece of work of Lev [38] in 1996 which shows that our estimates are rather delicate.

Suppose that the real number q is fixed, where $1 \leqslant q < \infty$. In view of the upper estimate in (1), clearly there exists sets $\mathscr{P}$ of N points such that the L_q-discrepancy satisfies the upper bound

$$\|D_{\mathcal{B}}(\mathscr{P})\|_q \ll_{k,q} (\log N)^{(k-1)/2}. \tag{9}$$

Let us treat the unit cube $U = [0,1]^k$ as a torus. For every $\mathbf{t} = (t_1, \ldots, t_k) \in [0,1]^k$, we now consider the translate

$$\mathscr{P} - \mathbf{t} = \{\mathbf{p} - \mathbf{t} : \mathbf{p} \in \mathscr{P}\}$$

of the point set $\mathscr{P}$. Then

$$\sup_{\mathbf{t}\in[0,1]^k} \|D_{\mathcal{B}}[\mathscr{P} - \mathbf{t}]\|_q \asymp_k \|D_{\mathcal{B}}[\mathscr{P}]\|_\infty, \tag{10}$$

where the implicit constants may depend on the dimension k, but not on q.

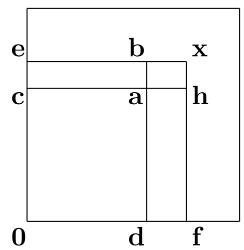
In view of the Great Open Problem, the inequality (10) tells us that the sharp upper bound (9) can be destroyed by a simple translation on the point set $\mathscr{P}$.

The original proof of Lev of the inequality (10) is an ingenious *tour de force* involving a number of different ideas. The main thrust is an induction argument up the dimensions. However, to make this work, one has to first consider the case $k = 1$, usually dismissed by most experts as trivial. Also, to make the induction work, it is necessary to introduce weights in order to hide some extra quantities that arise.

An elementary proof the inequalities (10), given later by Kolountzakis [36], is no less ingenious. We shall only discuss the case $k = 2$, as the proof generalizes naturally to higher dimensions. Suppose that $\mathscr{P}$ is a distribution of N points in the unit square $[0,1]^2$, treated as a torus. Note, first of all, that instead of shifting the set $\mathscr{P}$, we may equivalently shift the origin and the coordinate system and leave the set $\mathscr{P}$ in place. Suppose that

$$\|D_{\mathcal{B}}[\mathscr{P}]\|_\infty = M.$$

Then there exists a point $\mathbf{a} = (a_1, a_2) \in [0,1]^2$ such that $|D[\mathscr{P}; B(\mathbf{a})]| > M/2$. We assume that the set $\mathscr{A}$ of points $\mathbf{x} = (x_1, x_2) \in [0,1]^2$ such that $x_1 \geqslant a_1$ and $x_2 \geqslant a_2$ has measure at least $1/10$; the proof can be easily adjusted in other cases. For each such point $\mathbf{x}$, consider the following picture.



Let

$$\begin{aligned} R_1 &= \text{rectangle with vertices } \mathbf{0}, \mathbf{e}, \mathbf{x}, \mathbf{f}, \\ R_2 &= \text{rectangle with vertices } \mathbf{c}, \mathbf{e}, \mathbf{x}, \mathbf{h}, \\ R_3 &= \text{rectangle with vertices } \mathbf{d}, \mathbf{b}, \mathbf{x}, \mathbf{f}, \\ R_4 &= \text{rectangle with vertices } \mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{h}. \end{aligned}$$

Then

$$D[\mathscr{P}; B(\mathbf{a})] = D[\mathscr{P}; R_1] - D[\mathscr{P}; R_2] - D[\mathscr{P}; R_3] + D[\mathscr{P}; R_4],$$

and clearly

$$\max\{|D[\mathscr{P}; R_1]|, |D[\mathscr{P}; R_2]|, |D[\mathscr{P}; R_3]|, |D[\mathscr{P}; R_4]|\} > \frac{M}{8}.$$

Let $f(\mathbf{x}) = i$, where $i \in \{1,2,3,4\}$ and

$$|D[\mathscr{P}; R_i]| = \max\{|D[\mathscr{P}; R_1]|, |D[\mathscr{P}; R_2]|, |D[\mathscr{P}; R_3]|, |D[\mathscr{P}; R_4]|\},$$

with the convention that if there is more than one such value of i, then we choose the smallest such value. Clearly there exists one value $i^* \in \{1,2,3,4\}$ for which the set

$$\{\mathbf{x} \in \mathscr{A} : f(\mathbf{x}) = i^*\}$$

has measure at least $1/40$. Accordingly, we shift the origin to the point

$$\begin{cases} \mathbf{0}, & \text{if } i^* = 1, \\ \mathbf{c}, & \text{if } i^* = 2, \\ \mathbf{d}, & \text{if } i^* = 3, \\ \mathbf{a}, & \text{if } i^* = 4. \end{cases}$$

This implies that there exists $\mathbf{t} \in [0,1]^2$ such that $\|D[\mathscr{P} + \mathbf{t}]\|_1 \geqslant M/320$, and completes the proof.

Note that all the estimates in this classical setting are logarithmic in size in terms of the cardinality N of the point sets $\mathscr{P}$ in question. We sometimes refer to this as a *small discrepancy phenomenon*.

3. Some Work of Schmidt

There are many interesting discrepancy problems when we move away from the classical problem concerning aligned rectangular boxes anchored at the origin. The pioneering work in this direction is due to Schmidt [46, 47, 48] in 1969, using an integral equation technique and involving tilted rectangular boxes and balls as well as other geometric objects. The paper [48] is of particular interest. Let $U=[0,1]^k$, treated as a torus and with $k\geqslant 2$.

In the case when $\mathcal{B}$ is the collection of all rectangular boxes, we have the estimates

$$\inf_{|\mathscr{P}|=N}\|D_{\mathcal{B}}(\mathscr{P})\|_\infty \gg_\epsilon \begin{cases} N^{1/4-\epsilon}, & \text{if } k=2,\\ N^{1/3-\epsilon}, & \text{if } k\geqslant 3.\end{cases} \tag{11}$$

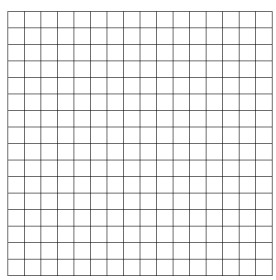
In the case when $\mathcal{B}$ is the collection of all balls, we have the estimate

$$\inf_{|\mathscr{P}|=N}\|D_{\mathcal{B}}(\mathscr{P})\|_\infty \gg_\epsilon N^{1/2-1/2k-\epsilon}. \tag{12}$$

Note that the estimates in these new settings are now powers of the cardinality N of the point sets $\mathscr{P}$ in question. We sometimes refer to these as a *large discrepancy phenomena.*

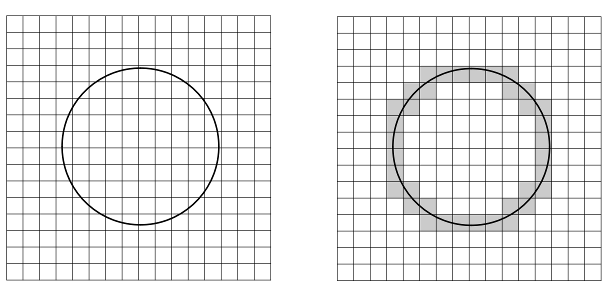
Indeed, apart from the term ϵ in the exponent, both estimates are essentially sharp, with the exception of (11) if $k>3$. We shall demonstrate this observation by Beck [4] in 1981 only in the special case $k=2$, as the argument generalizes to higher dimensions without any extra difficulties.

For simplicity, let us suppose that $N=M^2$ is a perfect square. Then we partition $U=[0,1]^2$ into N little squares in the usual way.



Let $\mathcal{S}$ denote the collection of all little squares, and we place one point anywhere in each such little square $S\in\mathcal{S}$, and denote by $\mathscr{P}$ the collection of all these points. This is a deterministic point set of precisely N points in U.

Now take any convex set $B\in\mathcal{B}$. This can be a tilted rectangle or a circular disc; the latter case is shown below on the left.



Clearly $D[\mathscr{P}; S \cap B] = 0$ whenever $S \cap B = \emptyset$ or $S \subseteq B$, and so

$$D[\mathscr{P}; B] = \sum_{S \in \mathcal{S}} D[\mathscr{P}; S \cap B] = \sum_{\substack{S \in \mathcal{S} \\ S \cap \partial B \neq \emptyset}} D[\mathscr{P}; S \cap B];$$

see the picture above on the right. The triangle inequality now leads to the estimate

$$|D[\mathscr{P}; B]| \leqslant \sum_{\substack{S \in \mathcal{S} \\ S \cap \partial B \neq \emptyset}} |D[\mathscr{P}; S \cap B]| \ll M = N^{1/2}.$$

This is rather crude, and clearly not good enough.

To get a better upper bound, we randomize the point set $\mathscr{P}$ by making the point in any little square $S \in \mathcal{S}$ a random point, uniformly distributed within that little square S and independently of the random points in the other little squares in $\mathcal{S}$. Applying large deviation techniques due to Bernstein–Chernoff or Hoeffding, this crude upper bound $N^{1/2}$ can then be converted to an upper bound of the form $N^{1/4}(\log N)^{1/2}$. The logarithmic factor represents the cost of using probability theory.

For slightly more details and related problems, see the survey article by Chen [23, Section 2].

4. Beck's Fourier Transform Technique

Let $\mathbb{T}^k = [0,1]^k$, treated as a torus and with $k \geqslant 2$. For any convex and compact set $B \subseteq [0,1]^k$, it is easy to see that $\mathbf{y} \in B + \mathbf{x}$ if and only if $\chi_{-B}(\mathbf{x} - \mathbf{y}) = 1$, where $-B = \{-\mathbf{y} : \mathbf{y} \in B\}$ and χ_{-B} denotes its characteristic function, and so

$$\begin{aligned} D[\mathscr{P}; B + \mathbf{x}] &= \sum_{\mathbf{p} \in \mathscr{P}} \chi_{-B}(\mathbf{x} - \mathbf{p}) - N \int_{\mathbb{T}^k} \chi_{-B}(\mathbf{x} - \mathbf{y}) \, \mathrm{d}\mu(\mathbf{y}) \\ &= \int_{\mathbb{T}^k} \chi_{-B}(\mathbf{x} - \mathbf{y}) \, (\mathrm{d}Z - N \mathrm{d}\mu)(\mathbf{y}). \end{aligned}$$

This can be summarized in the form

$$D = \chi_{-B} * (\mathrm{d}Z - N \mathrm{d}\mu). \tag{13}$$

In other words, under translation, discrepancy is a convolution of geometry and measure.

As lower bound estimates apply to arbitrary point sets $\mathscr{P}$, there is very limited information on the measure part of this convolution, and so we wish to concentrate on the geometry part. To separate the geometry part from the measure part, we apply Fourier transform. Then the convolution (13) becomes an ordinary product

$$\widehat{D} = \widehat{\chi_{-B}} \cdot (\mathrm{d}\widehat{Z - N}\mathrm{d}\mu)$$

of the Fourier transforms of the constituent parts.

This is the basis of Beck's Fourier transform technique, motivated by the work of Roth [44] in 1964 on irregularities of distribution of integer sequences.

Indeed, the similarity of the bounds (11) and (12) is no coincidence.

Let $U = [0,1]^k$, treated as a torus and with $k \geqslant 2$, and let A be a fixed convex and compact set in U satisfying some mild technical condition.

Suppose that the irregularity of a point set $\mathscr{P}$ in U is described in terms of the infinite collection

$$\mathcal{B} = \{A(\lambda, \tau, \mathbf{x}) : \lambda \in [0,1], \tau \in \mathcal{T}, \mathbf{x} \in [0,1]^k\}, \tag{14}$$

where $A(\lambda, \tau, \mathbf{x}) = \{\tau(\lambda\mathbf{y}) + \mathbf{x} : \mathbf{y} \in A\}$ denotes a similar copy obtained from the set A under a contraction λ, an orthogonal transformation τ and a translation $\mathbf{x}$, and where $\mathcal{B}$ is endowed with the integral geometric measure $\mathrm{d}B = \mathrm{d}\lambda\,\mathrm{d}\tau\,\mathrm{d}\mathbf{x}$.

Here we have the lower bounds

$$\inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_\infty \geqslant \inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_2 \gg_A N^{1/2-1/2k}, \tag{15}$$

due to Beck [6] in 1987.

A discussion of the special case $k = 2$ with a square A can be found in the survey by Chen [22, Section 2]. The special case $k = 2$ is discussed in generality in the lecture notes of Montgomery [41, Chapter 6]. There it is shown how estimates concerning the decay of the Fourier transform of the characteristic function of A lead to the lower bounds (15). Here Montgomery also discusses the special case $k = 2$ with a circular disc A of radius $1/2$. Note that rotation is irrelevant here. Note also that the Fourier transform of the characteristic function of A involves a Bessel function of the first kind, and so it does appear that contraction is essential. However, Montgomery can show that the contraction parameter λ can be restricted to a very small set. Indeed, he can show that the inequality

$$\int_{[0,1]^2} |D[\mathscr{P}; A(1,\mathbf{x})]|^2 \,\mathrm{d}\mathbf{x} + \int_{[0,1]^2} |D[\mathscr{P}; A(1/2,\mathbf{x})]|^2 \,\mathrm{d}\mathbf{x} \gg N^{1/2}$$

holds for every set $\mathscr{P}$ of N points in $[0,1]^2$. Here we have omitted reference to the unnecessary orthogonal transformation τ in our notation. Indeed, some average over contractions is necessary in view of the work of Parnovski and Sobolev [42]. See also the paper of Travaglini and Tupputi [55].

For an introduction to the relationship between the average decay of the Fourier transform and discrepancy theory, the interested reader is referred to the survey of Brandolini, Gigante and Travaglini [18].

Returning to our original problem, we also have the upper bounds

$$\inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_\infty \ll_A N^{1/2-1/2k}(\log N)^{1/2}, \tag{16}$$

obtained by Beck [4] in 1981, and

$$\inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_2 \ll_A N^{1/2-1/2k},$$

due to Beck and Chen [11] in 1990 and then improved to

$$\inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_q \ll_{A,q} N^{1/2-1/2k} \tag{17}$$

for any fixed positive real number q by Chen [21] in 2000.

Combining (15) and (17), it is clear that the L_q-discrepancy in this problem concerning all similar copies of a given convex and compact set in U is completely solved for every finite real number $q \geqslant 2$. However, comparing (15) and (16), we see that there is a gap in our knowledge for the L_∞-discrepancy in this problem.

Open Problem 3. *Let $U = [0,1]^k$, treated as a torus and with $k \geqslant 2$, and let A be a fixed convex and compact set in U satisfying some mild technical condition. Suppose that the set $\mathcal{B}$ is given by* (14)*. Does an estimate of the form*

$$\inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_\infty \ll_A N^{1/2-1/2k}$$

hold?

Suppose next that we no longer permit orthogonal transformation, and that the irregularity of a point set $\mathscr{P}$ in U is described in terms of the infinite collection

$$\mathcal{B} = \{A(\lambda, \mathbf{x}) : \lambda \in [0,1], \mathbf{x} \in [0,1]^k\}, \tag{18}$$

where $A(\lambda, \mathbf{x}) = \{\lambda\mathbf{y} + \mathbf{x} : \mathbf{y} \in A\}$ denotes a homothetic copy obtained from the set A under a contraction λ and a translation $\mathbf{x}$, and where $\mathcal{B}$ is endowed with the integral geometric measure $\mathrm{d}B = \mathrm{d}\lambda\,\mathrm{d}\mathbf{x}$. Then much less is known, and our limited knowledge is essentially restricted to the case $k = 2$, where we have a lower bound of the form

$$\inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_\infty \gg_A \max\{(\log N)^{1/2}, \xi(A,N)\}, \tag{19}$$

obtained by Beck [7] in 1988 from the corresponding estimate for the L_2-discrepancy. Here $\xi(A,N)$ is a function which depends on the boundary curve of the fixed set A. In particular, $\xi(A,N)$ is finite if A is a convex polygon, and $\xi(A,N) = N^{1/4}$ if A is a circular disc.

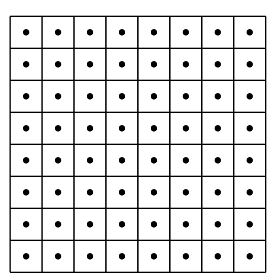
Open Problem 4 (Greater Open Problem)**.** *Let $U = [0,1]^k$, treated as a torus and with $k \geqslant 2$, and let A be a fixed convex and compact set in U satisfying some mild technical condition. Suppose that the set $\mathcal{B}$ is given by* (18)*.*

(i) (Generalization of the Bound (6)) *In the case $k = 2$, can the term* $(\log N)^{1/2}$ *in the estimate* (19) *be improved to* $\log N$*?*

(ii) *What can we say when $k \geqslant 3$?*

We complete this section by making a digression and discussing a result obtained in part by Fourier transform considerations.

Let $U = [0,1]^k$, treated as a torus and with $k \geqslant 1$. Suppose that the irregularity of a point set $\mathscr{P}$ in U is described in terms of the infinite collection $\mathcal{B}$ of all balls of diameter $1/2$. Let $\mathscr{P}$ denote the perfect square grid of $N = M^k$ points in U.



We now consider the quantity

$$\mathrm{DE}_k(M^k) = \int_{\mathcal{B}} |D[\mathscr{P}; B]|^2 \,\mathrm{d}B, \tag{20}$$

where the integral geometric measure $\mathrm{d}B$ is given by Lebesgue translation measure.

Next, let $\widetilde{\mathscr{P}}$ denote the random point set obtained by replacing each fixed point of $\mathscr{P}$ by a random point which is uniformly distributed in its own little cube and independently of any other random point in any other little cube. We consider the corresponding quantity

$$\mathrm{PR}_k(M^k) = \mathbb{E}\left(\int_{\mathcal{B}} |D[\widetilde{\mathscr{P}};B]|^2 \,\mathrm{d}B\right). \tag{21}$$

Which of the quantities (20) and (21) is smaller? We have the following surprising result, obtained by Chen and Travaglini [27] in 2009.

Suppose that $k \not\equiv 1 \bmod 4$. Then

- $\mathrm{DE}_2(M^2) \leqslant \mathrm{PR}_2(M^2)$ for all large M; and
- for all large k, $\mathrm{PR}_k(M^k) \leqslant \mathrm{DE}_k(M^k)$ for all large M.

Suppose that $k \equiv 1 \bmod 4$. Then

- for all large k, $\mathrm{PR}_k(M^k) \leqslant \mathrm{DE}_k(M^k)$ for infinitely many M;
- for all k, $\mathrm{DE}_k(M^k) \leqslant \mathrm{PR}_k(M^k)$ for infinitely many M; and
- $\mathrm{DE}_1(M) \leqslant \mathrm{PR}_1(M)$ for every M.

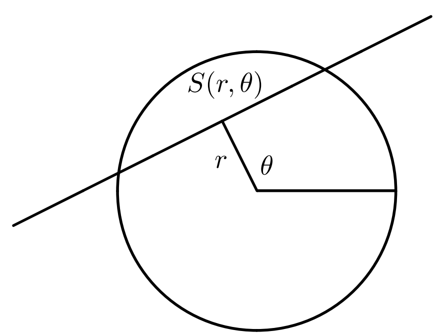
This is consistent with the work of Konyagin, Skriganov and Sobolev [37] in 2003 on lattice points in balls.

5. Roth's Disc Segment Problem

Let U be the circular disc in $\mathbb{R}^2$ of area 1 and centred at the origin. Suppose that the irregularity of a point set $\mathscr{P}$ in U is described in terms of the infinite collection

$$\mathcal{B} = \{S(r,\theta) : 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant r \leqslant \pi^{-1/2}\} \tag{22}$$

of disc segments in U.



A question of Roth concerns whether the quantity

$$\inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_\infty$$

is unbounded as a function of N.

Although this question never appeared in any of Roth's writings, it was recorded in Schmidt [48, Section I, last paragraph] as well as in Schmidt [52, Chapter II, §16].

To describe the results, it is useful to introduce the integral geometric measure $\mathrm{d}B = (2\pi^{1/2})^{-1}\,\mathrm{d}\theta\,\mathrm{d}r$, appropriately normalized so that the total measure equals unity.

Roth's question was answered in the affirmative by Beck [5] in 1983. Using his Fourier transform approach, suitably adapted, one can establish the lower bound

$$\inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_\infty \geqslant \inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_2 \gg N^{1/4}(\log N)^{-7/2}.$$

A stronger lower bound

$$\inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_\infty \geqslant \inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_2 \gg N^{1/4}, \tag{23}$$

via a new approach involving integral geometry, is due to Alexander [3] in 1990. On the other hand, it can be shown that

$$\inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_2 \leqslant \inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_\infty \ll N^{1/4}(\log N)^{1/2},$$

using the idea of Beck [4] in 1981. Here the factor $(\log N)^{1/2}$ arises for precisely the same reason as the corresponding factor in the estimate (16). However, there is no analogue of Open Problem 3 in this setting. Courtesy of an extraordinary piece of work by Matoušek [39] in 1995, it is now known that

$$\inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_2 \leqslant \inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_\infty \ll N^{1/4}. \tag{24}$$

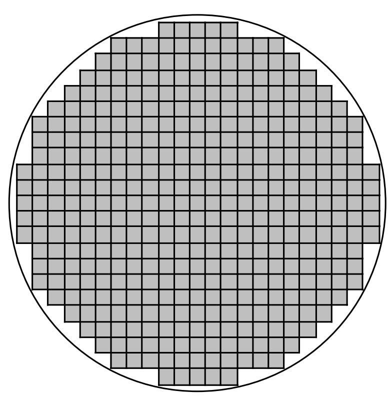
Combining the bounds (23) and (24), we see that the problem of the L_∞-discrepancy in this setting is completely solved, as is the problem of the L_q-discrepancy for any finite real number $q \geqslant 2$.

The situation is rather different if one studies the problem of the L_q-discrepancy in this setting when $1 \leqslant q < 2$.

Here, in particular when $q = 1$, the problem takes on some flavour of the classical discrepancy problem. Indeed, one can establish an upper bound of the form

$$\inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_1 \ll (\log N)^2, \tag{25}$$

as demonstrated by Beck and Chen [12] in 1993. The majority of the points of $\mathscr{P}$ come from a square grid.



The remaining points give rise to a *one-dimensional* discrepancy function along the boundary of the disc, and contribute only to the error terms. Thus for a fixed disc segment, the size of the discrepancy depends on the diophantine approximation properties of the slope of the boundary of the disc segment. What the estimate (25) tells us, therefore, is that in L_1-average, these properties of the slope are reasonably close to those of a badly approximable number, whereas the estimate (23) tells us that this is far from the case when we look at the corresponding L_2-average.

The argument can also be modified to show that

$$\inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_q \ll N^{(q-1)/2q}$$

for every real number q satisfying $1 < q \leqslant 2$. Note that the exponent drops from $1/4$ to 0 as q drops from 2 to 1.

Open Problem 5A. *Let U be the circular disc in $\mathbb{R}^2$ of area 1 and centred at the origin. Suppose that the irregularity of a point set $\mathscr{P}$ in U is described in terms of the infinite collection $\mathcal{B}$ given by* (22) *and endowed with the integral geometric measure* $\mathrm{d}B = (2\pi^{1/2})^{-1}\,\mathrm{d}\theta\,\mathrm{d}r$.

(i) *Improve the upper bound* (25) *if possible.*

(ii) *Find a lower bound for the problem of the L_1-discrepancy.*

An analogue of the Roth disc segment problem is the half plane problem in the unit cube. It is almost identical to the disc segment problem, except that we take $U = [-1/2, 1/2]^2$.

This problem can be extended to higher dimensions. Let $U = [-1/2, 1/2]^k$, with $k \geqslant 2$. Suppose that each half space $H(r, \mathbf{v})$ is characterized by its perpendicular distance r to the origin and its unit normal $\mathbf{v}$. Then we write

$$\mathcal{B} = \{S(r, \mathbf{v}) = H(r, \mathbf{v}) \cap U : \mathbf{v} \in S^{k-1}, r \geqslant 0\}, \tag{26}$$

where S^{k-1} denotes the surface of the sphere of radius 1 in $\mathbb{R}^k$, endowed with the integral geometric measure $\mathrm{d}B = \mathrm{d}\mathbf{v}\,\mathrm{d}r$, suitably normalized. In the special case $k = 2$, the bounds (23)–(25) remain valid. For arbitrary $k \geqslant 2$, an analogue of the bound (25) is given by

$$\inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_1 \ll_k (\log N)^k, \tag{27}$$

obtained by Chen and Travaglini [28] in 2011. A key ingredient in the argument is the divergence theorem which allows us to *climb* the dimensions.

Open Problem 5B. *Let $U = [-1/2, 1/2]^k$, with $k \geqslant 2$. Suppose that the irregularity of a point set $\mathscr{P}$ in U is described in terms of the infinite collection $\mathcal{B}$ given by* (26) *and endowed with the integral geometric measure* $\mathrm{d}B = \mathrm{d}\mathbf{v}\,\mathrm{d}r$*, suitably normalized.*

(i) *Improve the upper bound* (27) *if possible.*

(ii) *Find a lower bound for the problem of the L_1-discrepancy.*

Unfortunately, the technique of Chen and Travaglini has not so far been shown to work if we study the direct analogue of the Roth disc segment problem in higher dimensions.

Open Problem 6. *For every natural number $k \geqslant 3$, study the analogue of the Roth disc segment problem when U is the ball in $\mathbb{R}^k$ of volume* 1 *and centred at the origin.*

6. Problem of Convex Polygons

We all know that a convex polygon can be viewed as the intersection of finitely many half planes. This suggests that the idea surrounding the Roth disc segment problem can perhaps be transported over to this setting.

Let $U=[0,1]^2$, treated as a torus, and let A be a fixed convex polygon in U. The irregularity of a point set $\mathscr{P}$ in U is described in terms of the infinite collection

$$\mathcal{B}=\{A(\lambda,\tau,\mathbf{x}):\lambda\in[0,1],\tau\in\mathcal{T},\mathbf{x}\in[0,1]^2\}, \tag{28}$$

where $A(\lambda,\tau,\mathbf{x})=\{\tau(\lambda\mathbf{y})+\mathbf{x}:\mathbf{y}\in A\}$ denotes a similar copy obtained from the polygon A under a contraction λ, a rotation τ and a translation $\mathbf{x}$, and where $\mathcal{B}$ is endowed with the integral geometric measure $\mathrm{d}B=\mathrm{d}\lambda\,\mathrm{d}\tau\,\mathrm{d}\mathbf{x}$.

We see that this is a special case of a problem studied in Section 4. Corresponding to the bounds (15) and (17), we have

$$N^{1/4}\ll_{A,q}\inf_{|\mathscr{P}|=N}\|D_{\mathcal{B}}(\mathscr{P})\|_q\ll_{A,q}N^{1/4}$$

for every finite real number $q\geqslant 2$, so that the L_q-discrepancy in this problem is completely solved for these values of q.

As for the Roth disc segment problem, the situation is again rather different if one studies the problem of the L_q-discrepancy in this setting when $1\leqslant q<2$. Indeed, one can establish an analogous upper bound of the form

$$\inf_{|\mathscr{P}|=N}\|D_{\mathcal{B}}(\mathscr{P})\|_1\ll_A(\log N)^2, \tag{29}$$

as demonstrated by Beck and Chen [13] in 1993. The argument there can, as before, be modified to show that

$$\inf_{|\mathscr{P}|=N}\|D_{\mathcal{B}}(\mathscr{P})\|_{A,q}\ll N^{(q-1)/2q}$$

for every real number q satisfying $1<q\leqslant 2$.

Corresponding to Open Problem 5A, we have the following.

Open Problem 5C. *Let $U=[0,1]^2$, treated as a torus, and let A be a fixed convex polygon in U. Suppose that the irregularity of a point set $\mathscr{P}$ in U is described in terms of the infinite collection $\mathcal{B}$ given by* (28) *and endowed with the integral geometric measure* $\mathrm{d}B=\mathrm{d}\lambda\,\mathrm{d}\tau\,\mathrm{d}\mathbf{x}$.

(i) *Improve the upper bound* (29) *if possible.*

(ii) *Find a lower bound for the problem of the L_1-discrepancy.*

One can also study exceedingly difficult higher dimensional analogues.

Open Problem 5D. *Let $U=[0,1]^k$, treated as a torus and with $k\geqslant 2$, and let A be a fixed convex polytope in U. Suppose that the irregularity of a point set $\mathscr{P}$ in U is described in terms of the infinite collection*

$$\mathcal{B}=\{A(\lambda,\tau,\mathbf{x}):\lambda\in[0,1],\tau\in\mathcal{T},\mathbf{x}\in[0,1]^k\},$$

where $A(\lambda,\tau,\mathbf{x})=\{\tau(\lambda\mathbf{y})+\mathbf{x}:\mathbf{y}\in A\}$ denotes a similar copy obtained from the polytope A under a contraction λ, a rotation τ and a translation $\mathbf{x}$, and endowed with the integral geometric measure $\mathrm{d}B=\mathrm{d}\lambda\,\mathrm{d}\tau\,\mathrm{d}\mathbf{x}$. *What can one say about the L_1-discrepancy?*

Again, let $U=[0,1]^2$, treated as a torus, and let A be a fixed convex polygon in U. Suppose that we no longer permit rotation, and that the irregularity of a point set $\mathscr{P}$ in U is described in terms of the infinite collection

$$\mathcal{B}=\{A(\lambda,\mathbf{x}):\lambda\in[0,1],\mathbf{x}\in[0,1]^2\}$$

where $A(\lambda,\mathbf{x})=\{\lambda\mathbf{y}+\mathbf{x}:\mathbf{y}\in A\}$ denotes a homothetic copy obtained from the polygon A under a contraction λ and a translation $\mathbf{x}$, and where $\mathcal{B}$ is endowed with the integral geometric measure $\mathrm{d}B=\mathrm{d}\lambda\,\mathrm{d}\mathbf{x}$.

Note that if A is a rectangle, then this is somewhat analogous to the classical problem. Indeed, corresponding to the classical problem, we have the bound

$$\inf_{|\mathscr{P}|=N}\|D_{\mathcal{B}}(\mathscr{P})\|_2\ll_A(\log N)^{1/2},$$

due to Beck and Chen [14] in 1997. Funnily enough, this paper contains no new ideas, as all the major ingredients are known to Davenport and Roth, but not all of them to both.

We do not know whether the lower bound

$$\inf_{|\mathscr{P}|=N}\|D_{\mathcal{B}}(\mathscr{P})\|_2\gg_A(\log N)^{1/2},$$

analogous to Roth's classical result, the lower bound in (1) with $k=q=2$, holds.

Open Problem 7. *Let $U=[0,1]^k$, treated as a torus and with $k\geqslant 2$, and let A be a fixed convex polytope in U. Suppose that the irregularity of a point set $\mathscr{P}$ in U is described in terms of the infinite collection*

$$\mathcal{B}=\{A(\lambda,\mathbf{x}):\lambda\in[0,1],\mathbf{x}\in[0,1]^k\},$$

where $A(\lambda,\mathbf{x})=\{\lambda\mathbf{y}+\mathbf{x}:\mathbf{y}\in A\}$ denotes a homothetic copy obtained from the polytope A under a contraction λ and a translation $\mathbf{x}$, and endowed with the integral geometric measure $\mathrm{d}B=\mathrm{d}\lambda\,\mathrm{d}\mathbf{x}$. Is it true that

$$(\log N)^{(k-1)/2}\ll_A\inf_{|\mathscr{P}|=N}\|D_{\mathcal{B}}(\mathscr{P})\|_2\ll_A(\log N)^{(k-1)/2}?$$

7. Rotations of Rectangles

Throughout this section, let $U=[0,1]^2$. We first consider the problem concerning discrepancy of finite point sets in U with respect to various collections of convex polygons in U.

Suppose that $\mathcal{B}$ is the collection of all convex polygons in U with sides in Θ, where Θ is a fixed finite set of directions. Then

$$\log N\ll_\Theta\inf_{|\mathscr{P}|=N}\|D_{\mathcal{B}}(\mathscr{P})\|_\infty\ll_\Theta\log N. \tag{30}$$

The lower bound is due to Beck and Chen [10] in 1989, whereas the upper bound is due to Chen and Travaglini [25] in 2007.

We can expand the collection $\mathcal{B}$ in a number of ways. For instance, if $\mathcal{B}$ is the collection of all convex polygons in U of at most k sides, then

$$N^{1/4}\ll_k\inf_{|\mathscr{P}|=N}\|D_{\mathcal{B}}(\mathscr{P})\|_\infty\ll_k N^{1/4}(\log N)^{1/2}.$$

Here the lower bound is a simple consequence of the lower bound (15) when $k=2$, and the upper bound is again due to Chen and Travaglini [25] in 2007. On the other hand, if $\mathcal{B}$ is the collection of all convex convex polygons in U, then

$$N^{1/3}\ll\inf_{|\mathscr{P}|=N}\|D_{\mathcal{B}}(\mathscr{P})\|_\infty\ll N^{1/3}(\log N)^4.$$

Here the upper bound is due to Beck [8] in 1988, while the lower bound arises from an adaptation by Chen and Travaglini [25] in 2007 of an ingenious argument of Schmidt [50] in 1975.

Our original problem concerning convex polygons in U with sides in a finite set Θ has an analogous problem concerning finite rotations of rectangles. More precisely, suppose that $\mathcal{B}$ is the collection of all rectangles in U tilted by angles in a finite set Θ. Then the inequalities in (30) remain valid with this choice of $\mathcal{B}$.

A natural question is what happens if Θ is no longer finite. Some answers can be found in the work of Bilyk, Ma, Pipher and Spencer [16, 17] in 2011 and 2016. Using powerful results in diophantine approximation, it can be shown that

$$\inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_\infty \begin{cases} \ll_\Theta \log N, & \text{if } \Theta \text{ is a a finite set,} \\ \ll_\Theta (\log N)(\log\log N)^2, & \text{if } \Theta \text{ is a superlacunary set,} \\ \ll_\Theta (\log N)^3, & \text{if } \Theta \text{ is a lacunary sequence,} \\ \ll_\Theta (\log N)^{M+2}, & \text{if } \Theta \text{ is a lacunary set of order } M. \end{cases}$$

Furthermore, if Θ has upper Minkowski dimension $d \in [0,1)$, then

$$\inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_\infty \ll_{\Theta,\epsilon} N^{d/(d+1)+\epsilon}.$$

The following problem seems to be rather hard.

Open Problem 8. *Study the problem of discrepancy of points sets in $U = [0,1]^3$ with respect to polytopes in some suitable formulation.*

8. Cartesian Products

We complete this survey by discussing a problem motivated by some interesting work of Matoušek prompted by a question posed to him by the author in the first ever workshop on discrepancy theory in Kiel in 1998.

Consider first an example involving the well known classical discrepancy problem. Let $U_1 = [0,1]^k$, and let

$$\mathcal{B}_1 = \{B_1(\mathbf{x}) = [0,x_1) \times \ldots \times [0,x_k) : \mathbf{x} \in [0,1]^k\},$$

endowed with integral geometric measure $\mathrm{d}B_1 = \mathrm{d}\mathbf{x}$. We know that

$$(\log N)^{(k-1)/2} \ll_k \inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}_1}(\mathscr{P})\|_2 \ll_k (\log N)^{(k-1)/2}. \quad (31)$$

Next, let $U_2 = [0,1]^\ell$, and let

$$\mathcal{B}_2 = \{B_2(\mathbf{y}) = [0,y_1) \times \ldots \times [0,y_\ell) : \mathbf{y} \in [0,1]^\ell\},$$

endowed with integral geometric measure $\mathrm{d}B_2 = \mathrm{d}\mathbf{y}$. We know that

$$(\log N)^{(\ell-1)/2} \ll_\ell \inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}_2}(\mathscr{P})\|_2 \ll_\ell (\log N)^{(\ell-1)/2}. \quad (32)$$

Now let $U = U_1 \times U_2 = [0,1]^{k+\ell}$, and let

$$\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 = \{B_1(\mathbf{x}) \times B_2(\mathbf{y}) : \mathbf{x} \in [0,1]^k, \mathbf{y} \in [0,1]^\ell\},$$

endowed with integral geometric measure $\mathrm{d}B = \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}$. We know that

$$(\log N)^{(k+\ell-1)/2} \ll_{k,\ell} \inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_2 \ll_{k,\ell} (\log N)^{(k+\ell-1)/2}. \quad (33)$$

Note that the order of magnitude in the estimates in (33) is roughly the product of the order of magnitude of the estimates in (31) and the order of magnitude of the estimates in (32). Clearly both $\mathcal{B}_1$ and $\mathcal{B}_2$ play important roles.

Consider a second example. Let $U_1 = [0,1]^k$, treated as a torus, and let

$$\mathcal{B}_1 = \{A(\lambda,\tau,\mathbf{x}) : \lambda \in [0,1], \tau \in \mathcal{T}, \mathbf{x} \in [0,1]^k\},$$

endowed with integral geometric measure $\mathrm{d}B_1 = \mathrm{d}\lambda\,\mathrm{d}\tau\,\mathrm{d}\mathbf{x}$. We know from (15) and (17) that

$$N^{1/2-1/2k} \ll_A \inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}_1}(\mathscr{P})\|_2 \ll_A N^{1/2-1/2k}. \tag{34}$$

Next, let U_2, $\mathcal{B}_2$ and the integral geometric measure be the same as in the previous example, so that the estimates (32) remain valid. Now let $U = U_1 \times U_2 = [0,1]^{k+\ell}$, and let

$$\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 = \{A(\lambda,\tau,\mathbf{x}) \times B_2(\mathbf{y}) : \lambda \in [0,1], \tau \in \mathcal{T}, \mathbf{x} \in [0,1]^k, \mathbf{y} \in [0,1]^\ell\},$$

endowed with integral geometric measure $\mathrm{d}B_1 = \mathrm{d}\lambda\,\mathrm{d}\tau\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}$. We can show that

$$N^{1/2-1/2k} \ll_{A,\ell} \inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_2 \ll_{A,\ell} N^{1/2-1/2k}. \tag{35}$$

The lower bound follows from work of Beck [6] in 1987, while the upper bound is due to Beck and Chen [11] in 1990. We observe that the order of magnitude of the estimates in (34) and the order of magnitude of the estimates in (35) are identical, and the only contribution that the classical problem part of this cartesian product problem makes to the estimates in (35) is in the implicit constants. In other words, $\mathcal{B}_1$ dominates and $\mathcal{B}_2$ hardly matters.

To understand the situation a little better, we next consider a third example. Let $U_1 = [0,1]^2$, treated as a torus, and let $\mathcal{B}_1$ be the collection of all circular discs in U_1. Then it follows from (15) and (16) with $k = 2$ and noting that circular discs are invariant under rotation that

$$N^{1/4} \ll \inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}_1}(\mathscr{P})\|_\infty \ll N^{1/4}(\log N)^{1/2}. \tag{36}$$

Next, let $U_2 = [0,1]^4$, treated as a torus, and let $\mathcal{B}_2$ be the collection of all circular balls in U_2. Then it follows from (15) and (16) with $k = 4$ and noting that circular balls are invariant under orthogonal transformation that

$$N^{3/8} \ll \inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}_2}(\mathscr{P})\|_\infty \ll N^{3/8}(\log N)^{1/2}. \tag{37}$$

Finally, let $U = [0,1]^4 = U_1 \times U_1$, treated as a torus, and let $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_1$ be the collection in U of cartesian products of two circular discs in U_1. Then

$$\inf_{|\mathscr{P}|=N} \|D_{\mathcal{B}}(\mathscr{P})\|_\infty \ll_\epsilon N^{1/4+\epsilon}, \tag{38}$$

as shown by Matoušek [40] in 2000. Comparing the order of magnitude of the terms in (36) and (38), we see that the cartesian product of two copies of the 2-dimensional problem in U_1 does not produce any estimate that is substantially greater than the estimates produced by a single copy, and certainly nothing as large the estimates in (37) produced by the corresponding 4-dimensional problem in U_2. Indeed, in the paper of Matoušek, it is shown that, under certain conditions, the discrepancy estimates for a cartesian product problem is governed by the largest bound amongst the constituent parts.

Open Problem 9 (Matoušek's Problem)**.** *Try to obtain a better understanding concerning the discrepancy of cartesian products.*

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