# ON THE ERROR TERM OF THE PRIME NUMBER THEOREM AND THE DIFFERENCE BETWEEN THE NUMBER OF PRIMES IN THE RESIDUE CLASSES MODULO 4

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## 1. *Introduction*

Let $\pi(x)$ denote the number of primes not exceeding $x$. The Prime Number Theorem asserts that as $x \to \infty$,

$$\pi(x) \sim \operatorname{li} x,$$

where

$$\operatorname{li} x = \lim_{\eta \to 0+} \left( \int_0^{1-\eta} + \int_{1+\eta}^{x} \right) \frac{dy}{\log y}.$$

Equivalent to this is the relation that as $x \to \infty$,

$$\psi(x) \sim x,$$

where

$$\psi(x) = \sum_{n \leqslant x} \Lambda(n) = \sum_{p^m \leqslant x} \log p,$$

$p^m$ running through prime powers.

Let

$$P(x) = \operatorname{li} x - \pi(x) \quad \text{and} \quad R(x) = x - \psi(x).$$

The orders of magnitude of the functions $P(x)$ and $R(x)$ as $x \to \infty$ are closely connected with the distribution of the complex zeros of the Riemann zeta-function $\zeta(s)$. It is well-known that (see Ingham [**1**; p. 83, Theorem 30]) as $x \to \infty$,

$$P(x) = O(x^{\Theta} \log x) \quad \text{and} \quad R(x) = O(x^{\Theta} \log^2 x), \tag{1.1}$$

$\Theta$ being the upper bound of the real parts of the zeros of $\zeta(s)$. In addition, Littlewood [**2**] proved in 1914 that as $x \to \infty$,

$$P(x) = \Omega_{\pm}\left( x^{\frac{1}{2}} \frac{\log\log\log x}{\log x} \right) \quad \text{and} \quad R(x) = \Omega_{\pm}(x^{\frac{1}{2}} \log\log\log x). \tag{1.2}$$

However, there remains a gap between (1.1) and (1.2), even if the Riemann Hypothesis is true, i.e. $\Theta = \frac{1}{2}$.

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Let $\pi(x;q,a)$ denote the number of primes $p$ not exceeding $x$ and such that $p \equiv a(\bmod q)$. Let

$$\psi(x;q,a) = \sum_{\substack{n \leqslant x \\ n \equiv a(\bmod q)}} \Lambda(n).$$

Let

$$p(x) = \pi(x;4,3)-\pi(x;4,1) \quad \text{and} \quad r(x) = \psi(x;4,3)-\psi(x;4,1).$$

Then it is easily seen that $r(x) = -\psi(x,\chi)$, where $\chi$ is the non-principal character modulo 4, and

$$\psi(x,\chi) = \sum_{n \leqslant x} \chi(n)\Lambda(n).$$

Furthermore, the orders of magnitude of the functions $p(x)$ and $r(x)$ as $x \to \infty$ are closely connected with the distribution of the complex zeros of the Dirichlet $L$-function $L(s,\chi)$. On the one hand, we have that (see Prachar [3; p. 235, Satz 5.1]) as $x \to \infty$,

$$p(x) = O(x^{\theta}\log x) \quad \text{and} \quad r(x) = O(x^{\theta}\log^2 x), \tag{1.3}$$

$\theta$ being the upper bound of the real parts of the zeros of $L(s,\chi)$. On the other hand, it can be shown that as $x \to \infty$,

$$p(x) = \Omega_{\pm}\left(x^{\frac{1}{2}}\frac{\log\log\log x}{\log x}\right) \quad \text{and} \quad r(x) = \Omega_{\pm}(x^{\frac{1}{2}}\log\log\log x). \tag{1.4}$$

Again, there remains a gap between (1.3) and (1.4), even if the Riemann Hypothesis for $\chi$ is true, i.e. $\theta = \frac{1}{2}$.

In 1959, Shanks [4] conjectured that as $x \to \infty$,

$$\sum_{2 \leqslant n \leqslant x} \frac{P(n)n^{\frac{1}{2}}}{\pi(n)} \sim x \quad \text{and} \quad \sum_{2 \leqslant n \leqslant x} \frac{p(n)n^{\frac{1}{2}}}{\pi(n)} \sim x.$$

The object of this paper is to show that neither conjecture is true, and to prove the following theorems.

THEOREM 1. *Let*

$$\mathscr{R}(x) = \sum_{n \leqslant x} \frac{R(n)}{n^{\frac{1}{2}}}.$$

(A) *As* $x \to \infty$,

$$\mathscr{R}(x) = O(x^{\frac{1}{2}+\Theta}).$$

(B) *Suppose* $\Theta$ *is attained. Then as* $x \to \infty$,

$$\mathscr{R}(x) = \Omega_{\pm}(x^{\frac{1}{2}+\Theta}).$$

(C) *Suppose* $\alpha < \frac{1}{2}+\Theta$. *Then as* $x \to \infty$,

$$\mathscr{R}(x) = \Omega_{\pm}(x^{\alpha}).$$

THEOREM 2. *Let*

$$\mathscr{P}(x) = \sum_{2 \leqslant n \leqslant x} \frac{P(n)n^{\frac{1}{2}}}{\pi(n)}.$$

(A) *As* $x \to \infty$,

$$\mathscr{P}(x) = O(x^{\frac{1}{2}+\Theta}).$$

(B) *Suppose* $\Theta$ *is attained. If* $\Theta \neq \frac{1}{2}$, *i.e. if the Riemann Hypothesis is false, then as* $x \to \infty$,

$$\mathscr{P}(x) = \Omega_{\pm}(x^{\frac{1}{2}+\Theta}).$$

*If* $\Theta = \frac{1}{2}$, *i.e. if the Riemann Hypothesis is true, then as* $x \to \infty$,

$$\mathscr{P}(x) = x + \Omega_{\pm}(x).$$

(C) *Suppose* $\alpha < \frac{1}{2}+\Theta$. *Then as* $x \to \infty$,

$$\mathscr{P}(x) = \Omega(x^{\alpha}).$$

THEOREM 3. *Theorems* 1 *and* 2 *are true with* $\Theta$ *replaced by* $\theta$, *and* $\mathscr{R}(x)$ *and* $\mathscr{P}(x)$ *replaced by*

$$\mathscr{r}(x) = \sum_{n \leqslant x} \frac{r(n)}{n^{\frac{1}{2}}} \quad \text{and} \quad \mathscr{p}(x) = \sum_{2 \leqslant n \leqslant x} \frac{p(n)n^{\frac{1}{2}}}{\pi(n)}$$

*respectively. In particular, the analogue of Theorem* 2B *depends on the Riemann Hypothesis for* $\chi$, *where* $\chi$ *is the non-principal character modulo* 4.

It is easy to see that in view of Theorems 2B and 2C and their analogues, both conjectures are false.

We shall show in §4 that Theorems 2A and 2B are easy consequences of Theorems 1A and 1B. The proof of Theorem 1A is rather straightforward. To prove Theorems 1B, 1C and 2C, the method we use is an application of a theorem of Landau (Lemma C in §2) on Dirichlet integrals, and is basically similar to the method used by E. Schmidt to show that (see Ingham [1; pp. 90–92, Theorems 32–33]) as $x \to \infty$,

$$\psi(x) - x = \Omega_{\pm}(x^{\frac{1}{2}}),$$

and

$$\psi(x) - x = \Omega_{\pm}(x^{\Theta-\delta}) \quad \text{and} \quad \Pi(x) - \operatorname{li} x = \Omega_{\pm}(x^{\Theta-\delta}),$$

where $\delta$ is any fixed positive number, and where for $x \geqslant 2$,

$$\Pi(x) = \sum_{p^m \leqslant x} \frac{1}{m} = \sum_{2 \leqslant n \leqslant x} \frac{\Lambda(n)}{\log n} = \sum_{k=1}^{\infty}{}' \frac{\pi(x^{1/k})}{k}, \tag{1.5}$$

$\sum'$ signifying that there are only a finite number of non-zero terms in the sum.

In §§3–8, $\rho = \beta + i\gamma$ ($\beta, \gamma$ real) denotes a complex zero of $\zeta(s)$, where $s = \sigma + it$ ($\sigma, t$ real) denotes a complex variable. Also, for the sake of simplicity, we define, for $x \geqslant 2$,

$$\operatorname{Li} x = \int_2^x \frac{dy}{\log y}.$$

Then, for $x \geqslant 2$, we have $\operatorname{li} x - \operatorname{Li} x = \operatorname{li} 2$. It clearly suffices to prove Theorem 2 with $P(n)$ modified by replacing $\operatorname{li} n$ by $\operatorname{Li} n$.

In §9, we shall briefly indicate how Theorem 3 follows by slightly modifying the argument.

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## 2. *Some technical lemmas*

The first lemma is a convenient form of the formula for "partial summation" (see Ingham [1, p. 18, Theorem A]).

LEMMA A. *Let $\lambda_1, \lambda_2, \ldots$ be a real sequence which is non-decreasing and has the limit infinity, and let*

$$C(x) = \sum_{\lambda_n \leqslant x} c_n,$$

*where the $c_n$ may be real or complex, and the notation indicates a summation over the finite set of positive integers $n$ for which $\lambda_n \leqslant x$. Then if $Y \geqslant \lambda_1$, and $f(x)$ has a continuous derivative,*

$$\sum_{\lambda_n \leqslant Y} c_n f(\lambda_n) = -\int_{\lambda_1}^{Y} C(x) f'(x)dx + C(Y)f(Y).$$

In particular, suppose $X$ is a positive integer and $X \leqslant Y$. We put $\lambda_n = n$; and $c_n = 1$ for $n \geqslant X$ and $c_n = 0$ for $n < X$. On integrating by parts, one obtains the following lemma.

LEMMA B. *Suppose $f(x)$ has a continuous derivative in $[X, Y]$, where $X$ is a positive integer. Then, writing $\phi(x) = x - [x] - \frac{1}{2}$, we have*

$$\sum_{X \leqslant n \leqslant Y} f(n) = \int_X^Y f(x)dx + \tfrac{1}{2}f(X) - \phi(Y)f(Y) + \int_X^Y \phi(x)f'(x)dx.$$

The next lemma is a theorem of Landau on Dirichlet integrals (see Ingham [1; p. 88, Theorem H]), on which the proofs of Theorems 1B, 1C and 2C are based.

LEMMA C. *If $c(x)$, supposed real, is of constant sign for all sufficiently large $x$,* *then the real point* $s = \sigma_0$ *of the line of convergence of the Dirichlet integral*

$$\int_{x_0}^{\infty} \frac{c(x)}{x^s}\,dx\,,$$

*where* $x_0$ *is a constant, is a singularity of the function represented by the integral.*

The last lemma in this section is almost trivial, and we only include it here for the sake of convenience of reference later.

LEMMA D. *Suppose*

$$\sum_{x_0 \leqslant n \leqslant x} f(n) = O(x^{\delta}) \qquad \text{as } x \to \infty\,,$$

*and suppose* $g(n)$ *satisfies the following conditions:*

(i) $g(n) = o(1)$ *as* $n \to \infty$; *and*

(ii) $\sum_{x_0 \leqslant n \leqslant x} n^{\delta}|g(n)-g(n+1)| = o(x^{\delta})$ *as* $x \to \infty$.

*Then*

$$\sum_{x_0 \leqslant n \leqslant x} f(n)g(n) = o(x^{\delta}) \quad \text{as } x \to \infty\,.$$

## 3. *Proof of Theorem* 1A

Note, first of all, that

$$\sum_{n \leqslant x} \psi(n)n^{-\frac{1}{2}} = \sum_{m \leqslant x} \Lambda(m) \sum_{m \leqslant n \leqslant x} n^{-\frac{1}{2}}\,.$$

By applying Lemma B to $\sum_{n \leqslant x} n^{\frac{1}{2}}$ and $\sum_{m \leqslant n \leqslant x} n^{-\frac{1}{2}}$ separately, we see that as $x \to \infty$,

$$\sum_{n \leqslant x} n^{\frac{1}{2}} = \int_{2}^{x} y^{\frac{1}{2}}dy + O(x^{\frac{1}{2}})\,, \tag{3.1}$$

and

$$\sum_{n \leqslant x} \psi(n)n^{-\frac{1}{2}} = \sum_{m \leqslant x} \Lambda(m) \int_{m}^{x} y^{-\frac{1}{2}}dy + O\Big(\sum_{m \leqslant x} \Lambda(m)m^{-\frac{1}{2}}\Big)$$

$$= \int_{2}^{x} \psi(y)y^{-\frac{1}{2}}dy + O(x^{\frac{1}{2}}\log x)\,. \tag{3.2}$$

On combining (3.1) and (3.2), we have that as $x \to \infty$,

$$\sum_{n \leqslant x} \frac{R(n)}{n^{\frac{1}{2}}} = \int_2^x \frac{R(y)}{y^{\frac{1}{2}}} dy + O(x^{\frac{1}{2}} \log x), \tag{3.3}$$

and so Theorem 1A follows if we can show that as $x \to \infty$,

$$\int_2^x \frac{R(y)}{y^{\frac{1}{2}}} dy = \sum_\rho \frac{x^{\rho+\frac{1}{2}}}{\rho(\rho+\frac{1}{2})} + O(x^{\frac{1}{2}}) = O(x^{\frac{1}{2}+\Theta}), \tag{3.4}$$

where $\rho$ runs through all the complex zeros of $\zeta(s)$.

Let $x \geqslant 2$, and for $y$ satisfying $2 \leqslant y \leqslant x$, let

$$\psi_0(y) = \frac{\psi(y+0)+\psi(y-0)}{2}.$$

Then by Prachar [3; pp. 231–232, Satz 4.5], we have that

$$\int_2^x \frac{R(y)}{y^{\frac{1}{2}}} dy = \int_2^x \frac{y-\psi_0(y)}{y^{\frac{1}{2}}} dy = \int_2^x \sum_\rho \frac{y^{\rho-\frac{1}{2}}}{\rho} dy + \int_2^x \frac{\zeta'}{\zeta}(0) \frac{1}{y^{\frac{1}{2}}} dy + \tfrac{1}{2} \int_2^x \frac{1}{y^{\frac{1}{2}}} \log\left(1 - \frac{1}{y^{\frac{1}{2}}}\right) dy;$$

and if we write

$$\sum_\rho \frac{y^{\rho-\frac{1}{2}}}{\rho} = \sum_{|\gamma| \leqslant T} \frac{y^{\rho-\frac{1}{2}}}{\rho} + R_1(y, T), \tag{3.5}$$

then for $T \geqslant 2$,

$$R_1(y, T) \ll \frac{y^{\frac{1}{2}}}{T} \log^2 yT + \frac{\log y}{y^{\frac{1}{2}}} \qquad \text{(always)}, \tag{3.6}$$

$$R_1(y, T) \ll \begin{cases} \dfrac{y^{\frac{1}{2}}}{T}\left(\log^2 yT + \dfrac{\log y}{\xi}\right) & \text{(for non-integer } y), \\ \dfrac{y^{\frac{1}{2}}}{T} \log^2 yT & \text{(for integer } y); \end{cases} \tag{3.7}$$

where $\xi = \xi(y)$ is the distance of $y$ from the nearest integer.

By (3.7), it is easily seen that (3.5) is uniformly convergent over $[2, x]$, except possibly in the neighbourhood of integers, and by (3.6), also boundedly convergent over $[2, x]$. Therefore, it may be integrated term by term over $[2, x]$. Hence, as $x \to \infty$,

$$\int_2^x \frac{R(y)}{y^{\frac{1}{2}}} dy = \sum_\rho \frac{x^{\rho+\frac{1}{2}} - 2^{\rho+\frac{1}{2}}}{\rho(\rho+\frac{1}{2})} + O(x^{\frac{1}{2}}) = \sum_\rho \frac{x^{\rho+\frac{1}{2}}}{\rho(\rho+\frac{1}{2})} + O(x^{\frac{1}{2}}) = O(x^{\frac{1}{2}+\Theta}),$$

the last two steps being justified by the well-known fact that $\sum_{\gamma>0} \gamma^{-2}$ converges. Hence (3.4) follows.

## 4. *Derivation of Theorems* 2A *and* 2B *from Theorems* 1A *and* 1B

Let $Q(x) = \operatorname{Li} x - \Pi(x)$.

LEMMA 1. *As* $x \to \infty$,

$$\sum_{3 \leqslant n \leqslant x} \frac{Q(n)\log n}{n^{\frac{1}{2}}} - \sum_{n \leqslant x} \frac{R(n)}{n^{\frac{1}{2}}} = o(x^{\frac{1}{2}+\Theta}).$$

*Proof.* Let

$$R_1(x) = \int_2^x R(y)dy.$$

Then we deduce, as in the deduction of (3.4), that

$$R_1(x) = O(x^{1+\Theta}). \tag{4.1}$$

On the other hand, from (1.5) and by partial summation (Lemma A), we have

$$\Pi(x) = \sum_{2 \leqslant n \leqslant x} \frac{\Lambda(n)}{\log n} = \int_2^x \frac{\psi(y)}{y\log^2 y}dy + \frac{\psi(x)}{\log x}.$$

Also

$$\operatorname{Li} x = \int_2^x \frac{dy}{\log y} = \int_2^x \frac{y}{y\log^2 y}dy + \frac{x}{\log x} - \frac{2}{\log 2}.$$

Hence

$$Q(x) = \int_2^x \frac{R(y)}{y\log^2 y}dy + \frac{R(x)}{\log x} + O(1),$$

and so, on integrating by parts, we have

$$Q(n) - \frac{R(n)}{\log n} = \frac{R_1(n)}{n\log^2 n} - \int_2^n R_1(y)\frac{d}{dy}\left(\frac{1}{y\log^2 y}\right)dy + O(1). \tag{4.2}$$

It follows from (4.1) and (4.2) that

$$Q(n) - \frac{R(n)}{\log n} = O\left(\frac{n^{\Theta}}{\log^2 n}\right), \tag{4.3}$$

from which the lemma follows easily.

LEMMA 2. *As* $x \to \infty$,

$$\sum_{3 \leqslant n \leqslant x} \frac{Q(n)n^{\frac{1}{2}}}{\operatorname{Li} n} - \sum_{3 \leqslant n \leqslant x} \frac{Q(n)\log n}{n^{\frac{1}{2}}} = o(x^{\frac{1}{2}+\Theta}).$$

*Proof.* For $n \geqslant 3$,

$$\frac{Q(n)n^{\frac{1}{2}}}{\operatorname{Li} n} - \frac{Q(n)\log n}{n^{\frac{1}{2}}} = \frac{Q(n)\log n}{n^{\frac{1}{2}}}\left(\frac{n}{\operatorname{Li} n \log n} - 1\right).$$

In the notation of Lemma D, we put

$$f(n) = \frac{Q(n)\log n}{n^{\frac{1}{2}}} \quad \text{and} \quad g(n) = \frac{n}{\operatorname{Li} n \log n} - 1.$$

Then the difference in question is exactly $\sum_{3 \leqslant n \leqslant x} f(n)g(n)$. By Lemma 1 and Theorem 1A, the hypothesis of Lemma D on $f(n)$ is satisfied with $\delta = \frac{1}{2}+\Theta$. It remains to show that the hypotheses on $g(n)$ are satisfied. Now, for $n \geqslant 3$,

$$g(n) = \frac{1}{\operatorname{Li} n}\left(\frac{n}{\log n} - \operatorname{Li} n\right) = O\left(\frac{1}{\log n}\right) = o(1) \qquad \text{as } n \to \infty.$$

On the other hand, for $n \geqslant 3$,

$$|g(n) - g(n+1)| = \left|\frac{n}{\operatorname{Li} n \log n} - \frac{n+1}{\operatorname{Li}(n+1)\log(n+1)}\right|$$

$$\leqslant \frac{n}{\operatorname{Li} n}\left|\frac{1}{\log n} - \frac{1}{\log(n+1)}\right| + \frac{1}{\log(n+1)}\left|\frac{n}{\operatorname{Li} n} - \frac{n+1}{\operatorname{Li}(n+1)}\right|$$

$$= O\left(\frac{1}{\operatorname{Li} n \log^2 n} + \frac{n}{\operatorname{Li}^2 n \log^3 n}\right) = O\left(\frac{1}{n \log n}\right) \qquad \textit{as } n \to \infty.$$

Hence

$$n^{\delta}|g(n) - g(n+1)| = O\left(\frac{n^{\delta-1}}{\log n}\right) \qquad \text{as } n \to \infty.$$

The result now follows from Lemma D.

LEMMA 3. *As* $x \to \infty$,

$$\sum_{3 \leqslant n \leqslant x} \frac{P(n)n^{\frac{1}{2}}}{\mathrm{Li}\, n} - \sum_{3 \leqslant n \leqslant x} \frac{Q(n)n^{\frac{1}{2}}}{\mathrm{Li}\, n} \sim x .$$

*Proof.* For $n \geqslant 3$,

$$\frac{P(n)n^{\frac{1}{2}}}{\mathrm{Li}\, n} - \frac{Q(n)n^{\frac{1}{2}}}{\mathrm{Li}\, n} = \frac{\big(\Pi(n)-\pi(n)\big)n^{\frac{1}{2}}}{\mathrm{Li}\, n}.$$

Now, as $n \to \infty$,

$$\Pi(n)-\pi(n) = \sum_{k=2}^{\infty}{}' \frac{\pi(n^{1/k})}{k} = \tfrac{1}{2}\pi(n^{\frac{1}{2}})+O(n^{1/3}\log n) \sim \frac{n^{\frac{1}{2}}}{\log n} \tag{4.4}$$

by the Prime Number Theorem; therefore as $n \to \infty$,

$$\frac{\big(\Pi(n)-\pi(n)\big)n^{\frac{1}{2}}}{\mathrm{Li}\, n} \sim 1 .$$

The lemma now follows easily.

LEMMA 4. *As* $x \to \infty$,

$$\sum_{2 \leqslant n \leqslant x} \frac{P(n)n^{\frac{1}{2}}}{\pi(n)} - \sum_{3 \leqslant n \leqslant x} \frac{P(n)n^{\frac{1}{2}}}{\mathrm{Li}\, n} = o(x^{\frac{1}{2}+\Theta}) .$$

*Proof.* For $n \geqslant 3$,

$$\frac{P(n)n^{\frac{1}{2}}}{\pi(n)} - \frac{P(n)n^{\frac{1}{2}}}{\mathrm{Li}\, n} = \frac{P^2(n)n^{\frac{1}{2}}}{\pi(n)\,\mathrm{Li}\, n}. \tag{4.5}$$

If $\Theta < 1$, then by (1.1), we have that (4.5) is

$$O\left(\frac{n^{2\Theta-\frac{1}{2}}\log^4 n}{n}\right).$$

and the result follows since $2\Theta-\frac{1}{2} < \frac{1}{2}+\Theta$. If $\Theta = 1$, then by the inequality (see Ingham [**1**; p. 65, Theorem 23])

$$P(n) = O(ne^{-a\sqrt{\log n}}) , \tag{4.6}$$

where $a$ is an absolute constant, we have that (4.5) is

$$O\left(\frac{n^{\frac{1}{2}}\log^2 n}{e^{2a\sqrt{\log n}}}\right).$$

Lemma 4 follows.

On combining Lemmas 1, 2, 3 and 4 to effect the transition from $\sum_{n \leqslant x} n^{-\frac{1}{2}}R(n)$ to $\sum_{2 \leqslant n \leqslant x} P(n)n^{\frac{1}{2}}/\pi(n)$, we see that Theorems 1A and 1B yield Theorems 2A and 2B respectively.

## 5. *A stronger form of Theorem* 2C

In view of Theorem 2B, we can assume, without loss of generality, that the Riemann Hypothesis is false, and that $\Theta$ is not attained. We may further suppose that $1 < \alpha < \frac{1}{2}+\Theta$.

LEMMA 5. *Suppose as $x \to \infty$,*

$$\sum_{2 \leqslant n \leqslant x} \frac{P(n)n^{\frac{1}{2}}}{\pi(n)} = O(x^{\alpha}).$$

*Then as $x \to \infty$,*

$$\sum_{2 \leqslant n \leqslant x} \frac{Q(n)n^{\frac{1}{2}}}{\pi(n)} = O(x^{\alpha}).$$

*Proof.* Recall (4.4). We have

$$\Pi(n)-\pi(n) \sim \frac{n^{\frac{1}{2}}}{\log n}.$$

The lemma is now obvious in view of the Prime Number Theorem and the hypothesis that $\alpha > 1$.

LEMMA 6. *Suppose as $x \to \infty$,*

$$\sum_{2 \leqslant n \leqslant x} \frac{Q(n)n^{\frac{1}{2}}}{\pi(n)} = O(x^{\alpha}).$$

*Then as $x \to \infty$,*

$$\sum_{2 \leqslant n \leqslant x} \frac{Q(n)}{n^{\frac{1}{2}}} = o(x^{\alpha}).$$

*Proof.* We appeal to Lemma D. In the notation of Lemma D, we put

$$f(n) = \frac{Q(n)n^{\frac{1}{2}}}{\pi(n)} \quad \text{and} \quad g(n) = \frac{\pi(n)}{n}.$$

Then as $n \to \infty$, $g(n) = o(1)$. Also

$$|g(n)-g(n+1)| = \left|\frac{\pi(n)}{n}-\frac{\pi(n+1)}{n+1}\right|.$$

Suppose $n+1$ is not a prime, then as $n \to \infty$,

$$|g(n)-g(n+1)| = \pi(n)\left(\frac{1}{n}-\frac{1}{n+1}\right) = O\left(\frac{1}{n\log n}\right).$$

Suppose $n+1$ is a prime, then as $n \to \infty$,

$$|g(n)-g(n+1)| \leqslant \frac{1}{n+1}+\pi(n)\left(\frac{1}{n}-\frac{1}{n+1}\right) \leqslant \frac{1}{n}+O\left(\frac{1}{n\log n}\right).$$

Hence we have that as $x \to \infty$,

$$\sum_{n \leqslant x} n^{\alpha}|g(n)-g(n+1)| \leqslant O\left(\sum_{2 \leqslant n \leqslant x} \frac{n^{\alpha-1}}{\log n}\right)+x^{\alpha-1}\pi(x) = o(x^{\alpha}).$$

The result now follows from Lemma D.

In view of Lemmas 5 and 6, Theorem 2C will follow if we can show that as $x \to \infty$,

$$\sum_{2 \leqslant n \leqslant x} \frac{Q(n)}{n^{\frac{1}{2}}} = \Omega(x^{\alpha}).$$

In §8, we shall actually prove the stronger result that as $x \to \infty$,

$$\sum_{2 \leqslant n \leqslant x} \frac{Q(n)}{n^{\frac{1}{2}}} = \Omega_{\pm}(x^{\alpha}).$$

## 6. *Preparation for the proofs of Theorems* 1B, 1C *and* 2C

LEMMA 7. *For* $\operatorname{Re} s > \frac{1}{2}+\Theta$,

$$\sum_{n=1}^{\infty} \frac{R(n)}{n^{s+\frac{1}{2}}} - \int_{1}^{\infty} \frac{R(x)}{x^{s+\frac{1}{2}}}\,dx = A_1(s), \tag{6.1}$$

*where* $A_1(s)$ *is regular for* $\operatorname{Re} s > \frac{1}{2}$.

LEMMA 8. *For* $\operatorname{Re} s > \frac{1}{2}+\Theta$,

$$\sum_{n=2}^{\infty} \frac{Q(n)}{n^{s+\frac{1}{2}}} - \int_{2}^{\infty} \frac{Q(x)}{x^{s+\frac{1}{2}}}\,dx = A_2(s), \tag{6.2}$$

*where* $A_2(s)$ *is regular for* $\operatorname{Re} s > \frac{1}{2}$.

The proofs of Lemmas 7 and 8 are similar, so we shall only prove Lemma 8 here.

*Proof of Lemma* 8. As $X \to \infty$,

$$\sum_{2 \leqslant n \leqslant X} \Pi(n) n^{-(s+\frac{1}{2})} = \sum_{2 \leqslant n \leqslant X} \sum_{2 \leqslant m \leqslant n} \frac{\Lambda(m)}{\log m} n^{-(s+\frac{1}{2})}$$

$$= \sum_{2 \leqslant m \leqslant X} \frac{\Lambda(m)}{\log m} \sum_{m \leqslant n \leqslant X} n^{-(s+\frac{1}{2})}$$

$$= \sum_{2 \leqslant m \leqslant X} \frac{\Lambda(m)}{\log m} \left( \int_m^X x^{-(s+\frac{1}{2})} dx + \tfrac{1}{2} m^{-(s+\frac{1}{2})} - (X - [X] - \tfrac{1}{2}) X^{-(s+\frac{1}{2})} \right.$$

$$\left. + \int_m^X \left( \frac{d}{dx} x^{-(s+\frac{1}{2})} \right) (x - [x] - \tfrac{1}{2}) dx \right) \qquad \text{(by Lemma B)}$$

$$= \int_2^X \Pi(x) x^{-(s+\frac{1}{2})} dx + \tfrac{1}{2} \sum_{2 \leqslant m \leqslant X} \frac{\Lambda(m)}{\log m} m^{-(s+\frac{1}{2})} + O\left( \frac{\Pi(X)}{X^{\sigma+\frac{1}{2}}} \right)$$

$$+ \int_2^X \Pi(x) \left( \frac{d}{dx} x^{-(s+\frac{1}{2})} \right) (x - [x] - \tfrac{1}{2}) dx .$$

Also, as $X \to \infty$,

$$\sum_{2 \leqslant n \leqslant X} n^{-(s+\frac{1}{2})} \operatorname{Li} n = \int_2^X x^{-(s+\frac{1}{2})} \operatorname{Li} x dx + (X - [X] - \tfrac{1}{2}) X^{-(s+\frac{1}{2})} \operatorname{Li} X$$

$$+ \int_2^X \left( \frac{d}{dx} (x^{-(s+\frac{1}{2})} \operatorname{Li} x) \right) (x - [x] - \tfrac{1}{2}) dx \qquad \text{(by Lemma B)}$$

$$= \int_2^X x^{-(s+\frac{1}{2})} \operatorname{Li} x dx + O(X^{-(\sigma+\frac{1}{2})} \operatorname{Li} X)$$

$$+ \int_2^X \left( \frac{d}{dx} (x^{-(s+\frac{1}{2})} \operatorname{Li} x) \right) (x - [x] - \tfrac{1}{2}) dx .$$

It is easily checked that

$$\sum_{m=2}^{\infty} \frac{\Lambda(m)}{\log m} m^{-(s+\frac{1}{2})}, \qquad \int_2^{\infty} \Pi(x) \left( \frac{d}{dx} x^{-(s+\frac{1}{2})} \right) (x - [x] - \tfrac{1}{2}) dx$$

and

$$\int_2^\infty \left(\frac{d}{dx}(x^{-(s+\frac{1}{2})}\operatorname{Li}x)\right)(x-[x]-\tfrac{1}{2})dx$$

are all regular in $D = \{s : \operatorname{Re} s > \frac{1}{2}\}$. Furthermore, for every $\sigma > \frac{1}{2}$, we have that as $X \to \infty$,

$$\frac{\Pi(X)}{X^{\sigma+\frac{1}{2}}} \to 0 \quad \text{and} \quad \frac{\operatorname{Li} X}{X^{\sigma+\frac{1}{2}}} \to 0 .$$

Lemma 8 now follows on letting $X \to \infty$.

We have, by partial summation (Lemma A) that for real $s > \frac{3}{2}$,

$$\sum_{n \leqslant X} \frac{R(n)}{n^{s+\frac{1}{2}}} = s \int_1^X \frac{\sum_{n \leqslant x} n^{-\frac{1}{2}} R(n)}{x^{s+1}} dx + \frac{1}{X^s} \sum_{n \leqslant X} \frac{R(n)}{n^{\frac{1}{2}}}, \tag{6.3}$$

and

$$\sum_{2 \leqslant n \leqslant X} \frac{Q(n)}{n^{s+\frac{1}{2}}} = s \int_2^X \frac{\sum_{2 \leqslant n \leqslant x} n^{-\frac{1}{2}} Q(n)}{x^{s+1}} dx + \frac{1}{X^s} \sum_{2 \leqslant n \leqslant X} \frac{Q(n)}{n^{\frac{1}{2}}}. \tag{6.4}$$

Then on letting $X \to \infty$, and recalling (6.1) and (6.2), we see that (6.3) and (6.4) yield respectively

$$A_1(s) + \int_1^\infty \frac{R(x)}{x^{s+\frac{1}{2}}} dx = s \int_1^\infty \frac{\mathscr{R}(x)}{x^{s+1}} dx \qquad (s > \tfrac{3}{2}), \tag{6.5}$$

where

$$\mathscr{R}(x) = \sum_{n \leqslant x} \frac{R(n)}{n^{\frac{1}{2}}};$$

and

$$A_2(s) + \int_2^\infty \frac{Q(x)}{x^{s+\frac{1}{2}}} dx = s \int_2^\infty \frac{\mathscr{Q}(x)}{x^{s+1}} dx \qquad (s > \tfrac{3}{2}), \tag{6.6}$$

where

$$\mathscr{Q}(x) = \sum_{2 \leqslant n \leqslant x} \frac{Q(n)}{n^{\frac{1}{2}}}.$$

We have that (see Ingham [1; p. 18, equations (17) and (18)]) for real $s > \frac{3}{2}$,

$$(s-\tfrac{1}{2})\int_1^\infty \frac{\psi(x)}{x^{s+\frac{1}{2}}}\,dx = -\frac{\zeta'(s-\frac{1}{2})}{\zeta(s-\frac{1}{2})}, \tag{6.7}$$

and

$$(s-\tfrac{1}{2})\int_2^\infty \frac{\Pi(x)}{x^{s+\frac{1}{2}}}\,dx = \log\zeta(s-\tfrac{1}{2}). \tag{6.8}$$

On the other hand, for real $s > \frac{3}{2}$,

$$(s-\tfrac{1}{2})\int_2^\infty \frac{\mathrm{Li}\,x}{x^{s+\frac{1}{2}}}\,dx = \int_2^\infty \frac{dx}{x^{s-\frac{1}{2}}\log x} = \int_{(s-3/2)\log 2}^\infty \frac{e^{-y}}{y}\,dy \qquad (x^{s-3/2} = e^y)$$

$$= \int_{(s-3/2)\log 2}^1 \frac{dy}{y} + \int_{(s-3/2)\log 2}^1 \frac{e^{-y}-1}{y}\,dy + \int_1^\infty \frac{e^{-y}}{y}\,dy$$

$$= -\log(s-3/2)+g(s), \tag{6.9}$$

where $g(s)$ is an integral function.

Combining (6.5) and (6.7), we have that

$$\int_1^\infty \frac{\mathscr{R}(x)}{x^{s+1}}\,dx = A_3(s)+\frac{1}{s(s-\frac{1}{2})}\frac{\zeta'(s-\frac{1}{2})}{\zeta(s-\frac{1}{2})} + \frac{1}{s(s-3/2)} \qquad (s > 3/2), \tag{6.10}$$

where $A_3(s)$ is regular for $\mathrm{Re}\, s > \frac{1}{2}$. On the other hand, combining (6.6), (6.8) and (6.9), we have that

$$\int_2^\infty \frac{\mathscr{Q}(x)}{x^{s+1}}\,dx = A_4(s)-\frac{1}{s(s-\frac{1}{2})}\log\left((s-3/2)\zeta(s-\tfrac{1}{2})\right) \qquad (s > 3/2), \tag{6.11}$$

where $A_4(s)$ is regular for $\mathrm{Re}\, s > \frac{1}{2}$.

## 7. *Proof of Theorem* 1B

Suppose $\Theta$ is attained. Let $c(x)$ be such that

$$xc(x) = \mathscr{R}(x)+Cx^{\frac{1}{2}+\Theta}, \tag{7.1}$$

where $C$ is a positive constant. It follows from (6.10) that

$$\int_1^\infty \frac{c(x)}{x^s}\,dx = A_3(s)+\frac{1}{s}\left(\frac{1}{s-\frac{1}{2}}\frac{\zeta'(s-\frac{1}{2})}{\zeta(s-\frac{1}{2})}+\frac{1}{s-\frac{3}{2}}\right)+\frac{C}{s-(\frac{1}{2}+\Theta)} = F(s), \text{ say}, \quad (s > \tfrac{3}{2}). \tag{7.2}$$

Let $\sigma_0$ be the abscissa of convergence of the Dirichlet integral in (7.2). Then the integral represents a single-valued branch of $F(s)$ regular in $\sigma > \sigma_0$. Suppose, if possible, that $c(x) \geqslant 0$ for all $x \geqslant X$. Then since $F(s)$ has a singularity at $s = \frac{1}{2}+\Theta$ but at no point on the real axis to the right of this, both $A_3(s)$ and

$$\left(\frac{1}{s-\frac{1}{2}}\frac{\zeta'(s-\frac{1}{2})}{\zeta(s-\frac{1}{2})}+\frac{1}{s-3/2}\right)$$

being regular along this stretch (including at $s = 3/2$), it follows from Lemma C that $\sigma_0 = \frac{1}{2}+\Theta$, and so $\sigma = \frac{1}{2}+\Theta$ is the line of convergence of the Dirichlet integral, so that (7.2) is valid for $\sigma > \frac{1}{2}+\Theta$. Hence, for $\sigma > \frac{1}{2}+\Theta$,

$$|F(\sigma+it)| \leqslant \int_1^X \frac{|c(x)|}{x^\sigma}\,dx+\int_X^\infty \frac{c(x)}{x^\sigma}\,dx = \int_1^X \frac{|c(x)|-c(x)}{x^\sigma}\,dx+F(\sigma)$$

$$\leqslant 2\int_1^X \frac{|c(x)|}{x^{\frac{1}{2}+\Theta}}\,dx+F(\sigma) = K+F(\sigma), \tag{7.3}$$

where $K$ is independent of $\sigma$ and $t$. Take $t = \gamma_1$, where among the zeros of $\zeta(s)$ with $\sigma = \Theta, \Theta+i\gamma_1$ is the one with least positive imaginary part; and let $m$ be its multiplicity. On multiplying both sides of (7.3) by $\sigma-(\frac{1}{2}+\Theta)$, and letting $\sigma \to (\frac{1}{2}+\Theta)+0$, we have

$$\frac{m}{|(\frac{1}{2}+\Theta+i\gamma_1)(\Theta+i\gamma_1)|} \leqslant C.$$

We now choose $C$ so that

$$0 < C < \frac{m}{|(\frac{1}{2}+\Theta+i\gamma_1)(\Theta+i\gamma_1)|}.$$

Then the supposition that $c(x) \geqslant 0$ for all $x \geqslant X$ leads to a contradiction, so $c(x) < 0$ for arbitrarily large $x$, i.e.

$$\mathscr{R}(x) < -Cx^{\frac{1}{2}+\Theta} \text{ for arbitrarily large } x.$$

Replacing $C$ by $-C$ in (7.1), the above argument yields

$$\mathscr{R}(x) > Cx^{\frac{1}{2}+\Theta} \text{ for arbitrarily large } x.$$

## 8. *Proof of Theorems* 1C *and* 2C

In view of Theorems 1B and 2B, we assume, without loss of generality, that $\Theta \neq \frac{1}{2}$, $1 < \alpha < \frac{1}{2}+\Theta$, and that $\Theta$ is not attained.

We write $W_j(x)$ $(j = 1, 2)$ for $\mathscr{R}(x)$ and $\mathscr{Q}(x)$ respectively. Also, note that the right-hand sides of (6.10) and (6.11) are regular in the stretch $s > \alpha$ of the real axis, including at $s = 3/2$. We call them $H_j(s)$ $(j = 1, 2)$ respectively. Then (6.10) and (6.11) are of the form

$$\int_j^\infty \frac{W_j(x)}{x^{s+1}}\,dx = H_j(s) \qquad (s > 3/2),$$

where $j = 1, 2$ respectively, and where $H_j(s)$ is regular in the stretch $s > \alpha$ on the real axis.

Let $c_j(x)$ be such that

$$xc_j(x) = W_j(x) - x^\alpha. \tag{8.1}$$

Then

$$\int_j^\infty \frac{c_j(x)}{x^s}\,dx = H_j(s) - \frac{j^{\alpha-s}}{s-\alpha} = G_j(s), \text{ say,} \qquad (s > 3/2). \tag{8.2}$$

Let $\sigma_0$ be the abscissa of convergence of the Dirichlet integral in (8.2). Then the integral represents a single-valued branch of $G_j(s)$ regular in $\sigma > \sigma_0$. It is clear from (8.2) and either (6.10) or (6.11) that no such branch could exist if the half-plane $\sigma > \sigma_0$ contained any zeros of $\zeta(s-\frac{1}{2})$. Hence we must have $\sigma_0 \geqslant \frac{1}{2}+\Theta$. On the other hand, $G_j(s)$ has no singularities in the stretch $s > \alpha$ of the real axis, $H_j(s)$ being regular along this stretch. In particular, we have $\sigma_0 \geqslant \frac{1}{2}+\Theta > \alpha$, so $s = \sigma_0$ is not a singularity of $G_j(s)$. It follows from Lemma C that we cannot have, for example, $c_j(x) \leqslant 0$ for $x \geqslant X_j$. Hence for $j = 1, 2$,

$$W_j(x) > x^\alpha \text{ for arbitrarily large } x.$$

On replacing (8.1) by

$$xc_j(x) = W_j(x) + x^\alpha,$$

the above argument yields for $j = 1, 2$,

$$W_j(x) < -x^\alpha \text{ for arbitrarily large } x.$$

## 9. *Theorem* 3

First of all, recall that $r(x) = -\psi(x, \chi)$, where $\chi$ is the non-principal character modulo 4, and

$$\psi(x, \chi) = \sum_{n \leqslant x} \chi(n)\Lambda(n);$$

and so, by analogy with (3.3), we have

$$\sum_{n \leqslant x} \frac{r(n)}{n^{\frac{1}{2}}} = \int_2^x \frac{r(y)}{y^{\frac{1}{2}}}\, dy + O(x^{\frac{1}{2}} \log x).$$

The analogue of Theorem 1A follows from Prachar [3; pp. 228–229, Satz 4.4] and the well-known fact that $\sum_{\gamma > 0} \gamma^{-2}$ converges, where $\rho = \beta + i\gamma$ ($\beta, \gamma$ real) are now the complex zeros of $L(s, \chi)$.

The analogues of Theorems 2A and 2B can be derived from the analogues of Theorems 1A and 1B. By defining $q(x) = -\Pi(x, \chi)$, where, for $x \geqslant 2$,

$$\Pi(x, \chi) = \sum_{p^m \leqslant x} \frac{\chi(p^m)}{m} = \sum_{2 \leqslant n \leqslant x} \frac{\chi(n)\Lambda(n)}{\log n},$$

we see that

$$q(n) - \frac{r(n)}{\log n} = O\left(\frac{n^{\theta}}{\log^2 n}\right),$$

and that

$$\begin{aligned} p(n) - q(n) &= \sum_{k=2}^{\infty}{}' \frac{\pi(n^{1/k}; 4, 1) + (-1)^k \pi(n^{1/k}; 4, 3)}{k} \\ &= \tfrac{1}{2}\pi(n^{\frac{1}{2}}; 4, 1) + \tfrac{1}{2}\pi(n^{\frac{1}{2}}; 4, 3) + O(n^{1/3} \log n) \\ &= \tfrac{1}{2}\pi(n^{\frac{1}{2}}) + O(n^{1/3} \log n) \sim \frac{n^{\frac{1}{2}}}{\log n}, \end{aligned}$$

which is analogous to (4.5). For the analogue of Lemma 4, we apply (1.3) and (4.6).

The analogues of Theorems 1B, 1C and 2C are established, noting that $L(s, \chi)$ is regular and different from 0 at $s = 1$, and that for real $s > 3/2$,

$$(s - \tfrac{1}{2}) \int_1^{\infty} \frac{\psi(x, \chi)}{x^{s+\frac{1}{2}}}\, dx = -\frac{L'(s - \frac{1}{2}, \chi)}{L(s - \frac{1}{2}, \chi)},$$

and

$$(s - \tfrac{1}{2}) \int_2^{\infty} \frac{\Pi(x, \chi)}{x^{s+\frac{1}{2}}}\, dx = \log L(s - \tfrac{1}{2}, \chi).$$

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