ON IRREGULARITIES OF DISTRIBUTION

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§1. *Introduction.* Let $U_0 = [0, 1)$ and $U_1 = (0, 1]$. Suppose we have a distribution $\mathscr{P}$ of N points in U_0^{k+1}, where, for $k \geqslant 1, U_0^{k+1}$ is the unit cube consisting of the points $\mathbf{y} = (y_1, \ldots, y_{k+1})$ with $0 \leqslant y_i < 1$ $(i = 1, \ldots, k+1)$. For $\mathbf{x} = (x_1, \ldots, x_{k+1})$ in U_1^{k+1}, let $B(\mathbf{x})$ denote the box consisting of all $\mathbf{y}$ such that $0 \leqslant y_i < x_i$ $(i = 1, \ldots, k+1)$, and let $Z[\mathscr{P}; B(\mathbf{x})]$ denote the number of points of $\mathscr{P}$ which lie in $B(\mathbf{x})$. Write

$$D[\mathscr{P}; B(\mathbf{x})] = Z[\mathscr{P}; B(\mathbf{x})] - Nx_1 \ldots x_{k+1}.$$

The irregularity of the distribution $\mathscr{P}$ can be measured in a number of ways by the behaviour of the function $D[\mathscr{P}; B(\mathbf{x})]$. One may consider the L^W-norm

$$\|D(\mathscr{P})\|_W = \left(\int_{U_1} \ldots \int_{U_1} |D[\mathscr{P}; B(\mathbf{x})]|^W dx_1 \ldots dx_{k+1} \right)^{1/W}.$$

Roth [3] obtained a lower bound for the L^2-norm, and Schmidt [7] established the following generalization, Roth's result being the special case $W = 2$.

Theorem 1. (Schmidt [7]). *For every $W > 1$, there exists a positive number $c_1(k, W)$, depending only on k and W, such that*

$$\|D(\mathscr{P})\|_W > c_1(k, W)(\log N)^{\frac{1}{2}k}.$$

Roth's lower bound for the L^2-norm has been shown to be sharp, apart from the value of the constant. This was established in the special cases $k = 1$ and $k = 2$ by Davenport [1] and Roth [5] respectively, and more recently for arbitrary k by Roth [6]. Different proofs for the case $k = 1$ were given by Vilenkin [8], Halton and Zaremba [2] and Roth [4].

The object of this paper is to show that Schmidt's lower bound, given in Theorem 1 above, is also sharp, apart from the value of the constant. Accordingly, we shall prove the following theorem.

Theorem 2. *Let $W > 0$. For a suitable number $c_2(k, W)$, depending only on k and W, there exists, corresponding to every natural number $N \geqslant 2$, a distribution $\mathscr{P}$, which may depend on W, of N points in U_0^{k+1} such that*

$$\|D(\mathscr{P})\|_W < c_2(k, W)(\log N)^{\frac{1}{2}k}.$$

The two-dimensional case, $k = 1$, of Theorem 2 can easily be deduced from the argument in Roth [4]. On the other hand, since the L^W-norm increases with W, the case $W \leqslant 2$ of Theorem 2 follows immediately as a consequence of Roth [6]. Our result is therefore new only when $k \geqslant 2$, $W > 2$.

Our method is, as in Roth [6], based on the consideration of "translations" of the Hammersley (Halton) sequence, an average with respect to such translations being taken over a "long" interval. In the application of the Main Lemma to deduce Theorem 2 (see §7), the set $\Omega = \Omega^{(1)}$ is chosen in terms of the Hammersley (Halton) sequence† exactly as in [6]. With this choice of Ω, Roth's "Basic Lemma" in [6] is the case $W = 2$ of our Main Lemma. However, to prove the Main Lemma, new ideas are required to overcome difficulties that did not arise in the case $W = 2$. The proof of the Main Lemma is new, and is based on the use of induction which was not needed in the special case $W = 2$ dealt with by Roth in [6]. To make the induction work, we have to make use of sequences (sets) (see §3) which are of a more general nature than the Hammersley (Halton) sequence, whilst retaining its relevant properties. We therefore have to introduce a wider class of sets Ω to give a formulation of our Main Lemma suitable for proof by induction. We also remark that the derivation of Lemma 5, a key identity in the proof of the Main Lemma, requires an elaboration of the idea underlying Roth's proof of his Basic Lemma (see (24) and (36)).

I am greatly indebted to Professor Roth for his constant encouragement and helpful suggestions, and for giving me manuscripts of his papers [5] and [6]. I also thank the referee for his valuable comments.

§2. *Notation.* By an interval I, we shall mean a half-open interval of the type $[\alpha_1, \alpha_2)$, while $I_1 \times I_2$ denotes the Cartesian product of I_1 and I_2.

We use R to denote a residue class. In particular, $R(m, q)$ denotes the residue class of integers congruent to m modulo q. For any real number t, we denote by $t+R$ the set $\{t+n : n \in R\}$, and write

$$F[t+R\,; I] = Z[t+R\,; I] - q^{-1}\ell(I), \tag{1}$$

where $Z[t+R\,; I]$ denotes the number of elements of $t+R$ falling into I, q is the modulus of the residue class R, and $\ell(I)$ is the length of I. It is obvious that

$$|F[t+R\,; I]| \leqslant 1 \tag{2}$$

always.

Throughout, $\mathbb{Z}$ denotes the set of all integers.

§3. *Generalization of the idea of the Hammersley* (*Halton*) *sequence.* Let h be a positive integer, and let $p_1, \ldots, p_k$ be distinct primes. For any integer n satisfying $0 \leqslant n < (p_1 \ldots p_k)^h$ and for every $j = 1, \ldots, k$, write

$$n = \sum_{v=0}^{\infty}{}' a_{j,v} p_j^v \qquad (0 \leqslant a_{j,v} < p_j),$$

where $\sum'$ signifies that there is only a finite number of non-zero terms in the sum. In

† In fact, any choice of $\Omega^{(1)}$ which satisfies the Main Lemma would do. However, the value of t^* in (73) would then depend on our choice of $\Omega^{(1)}$.

is clear that the integers $a_{j,v}$ are uniquely determined by n. We write

$$X_j(n) = p_j^{-1} \sum_{v=0}^{\infty} a_{j,v} p_j^{-v} .$$

Then $X_j(n)$ lies in U_0. Let

$$\mathbf{X}(n) = (X_1(n), \ldots, X_k(n)) .$$

Then the vectors

$$\mathbf{X}(0), \ldots, \mathbf{X}((p_1 \ldots p_k)^h - 1)$$

are the first $(p_1 \ldots p_k)^h$ terms of the Hammersley (Halton) sequence. We extend the range of definition of $\mathbf{X}(n)$ over the set $\mathbb{Z}$ of all integers by periodicity so as to ensure that

$$\mathbf{X}(n + (p_1 \ldots p_k)^h) = \mathbf{X}(n)$$

for every integer n. In [**6**], Roth considered the set

$$\Omega'(t) = \{(\mathbf{X}(n), n+t) : n \in \mathbb{Z}\} , \tag{3}$$

where $(\mathbf{X}(n), n+t)$ stands for the point $(X_1(n), \ldots, X_k(n), n+t)$ in $(k+1)$-dimensional space.

In this paper, we need to consider sets which are of a more general nature than (3). Let h be a positive integer, and let $p_1, \ldots, p_k$ be distinct primes. We define a q-set of class h with respect to the primes $p_1, \ldots, p_k$. Let $1 \leqslant j \leqslant k$.

Definition. Let $0 \leqslant s \leqslant h$. An interval $[\alpha_1, \alpha_2)$ is said to be an elementary p_j-type interval of order s if α_1, α_2 are consecutive integer multiples of p_j^{-s}, *and* $[\alpha_1, \alpha_2)$ is contained in U_0.

We reserve the symbol J for elementary intervals.

By an association of order h with respect to the prime p_j, we shall mean the following. We associate each elementary p_j-type interval of order s with a residue class modulo p_j^s in such a way that if J_1 and J_2 are elementary p_j-type intervals, of orders s_1 and s_2 respectively, where $0 \leqslant s_2 \leqslant s_1 \leqslant h$, and if R_1 and R_2 are respectively their associated residue classes, then

$$J_1 \subset J_2 \quad \Leftrightarrow \quad R_1 \subset R_2 . \tag{4}$$

Since either $J_1 \subset J_2$ or $J_1 \cap J_2 = \varnothing$, and a similar relation applies to R_1 and R_2, we see, in particular, that this association sets up a one-to-one correspondence between the collection of elementary p_j-type intervals of order s and the collection of residue classes modulo p_j^s for every s satisfying $0 \leqslant s \leqslant h$.

Let Q be a residue class modulo q, where $(q, p_1 \ldots p_k) = 1$, and suppose $\mathscr{X} = \{\mathbf{x}(n) : n \in Q\}$ is a set, where for each $n \in Q$, $\mathbf{x}(n) = (x_1(n), \ldots, x_k(n))$ is a point in U_0^k. Consider the set

$$\Omega(Q, \mathscr{X}) = \{(\mathbf{x}(n), n) : n \in Q\} . \tag{5}$$

We are concerned with sets of the form (5) and having a very special property. Our next definition is motivated by a special property of $\Omega'(0)$, where $\Omega'(t)$ is the set (3).

Definition. Suppose $\mathscr{X}$ is such that, for every integer $j = 1, \ldots, k$, we can define an association between elementary p_j-type intervals and residue classes modulo p_j^s $(0 \leqslant s \leqslant h)$ as follows. Given any residue class R modulo p_j^s, where $0 \leqslant s \leqslant h$, there exists an elementary p_j-type interval J of order s such that $\{x_j(n) : n \in Q \cap R\} \subset J$; in other words, the j-th coordinates of those points $\mathbf{x}(n)$, for which n belongs to a residue class modulo p_j^s, as well as Q, all fall into the same elementary p_j-type interval of order s. We associate J with R. Suppose further that, for every $j = 1, \ldots, k$, the corresponding association is of order h with respect to the prime p_j; in other words, that for every $j = 1, \ldots, k$, the set $\mathscr{X}$ sets up an association of order h with respect to the prime p_j. Then we say that $\Omega(Q, \mathscr{X})$ is a q-set of class h with respect to the primes $p_1, \ldots, p_k$.

We note that, given any q-set $\Omega(Q, \mathscr{X})$ of class h with respect to the primes $p_1, \ldots, p_k$, the association of order h with respect to each prime is uniquely determined.

For the sake of simplicity, since it is important only to bear in mind the modulus q of the residue class Q when considering sets of the type $\Omega(Q, \mathscr{X})$, we write $\Omega^{(q)}$ for $\Omega(Q, \mathscr{X})$. We shall consider sets of the form

$$\Omega^{(q)}(t) = \Omega(Q, \mathscr{X}; t) = \{(\mathbf{x}(n), n+t) : n \in Q\}. \tag{6}$$

Note that $\Omega^{(q)}(0)$ is the set (5).

Definition. Suppose $\Omega^{(q)}$ is a q-set of class h with respect to the primes $p_1, \ldots, p_k$; and suppose t is any real number. We say that the set $\Omega^{(q)}(t)$ is a translated q-set of class h with respect to the same primes.

It is clear that the set (3) given by the Hammersley (Halton) sequence is a translated 1-set of class h with respect to the primes $p_1, \ldots, p_k$. Furthermore, if Q is a residue class modulo q, where $(q, p_1 \ldots p_k) = 1$, the subset $\{(\mathbf{X}(n), n+t) : n \in Q\}$ of (3) is a translated q-set of class h with respect to the primes $p_1, \ldots, p_k$. On the other hand, the following lemmas are obvious.

LEMMA 1. *Suppose $\Omega^{(q)}(t)$ is a translated q-set of class h with respect to the primes $p_1, \ldots, p_k$. Then, for every s satisfying $1 \leqslant s \leqslant h$, $\Omega^{(q)}(t)$ is also a translated q-set of class s with respect to the same primes.*

LEMMA 2. *Suppose $\Omega^{(1)}(t) = \{(\mathbf{x}(n), n+t) : n \in \mathbb{Z}\}$ is a translated 1-set of class h with respect to the primes $p_1, \ldots, p_k$. Then, for every $j = 1, \ldots, k$, the set*

$$\{(x_1(n), \ldots, x_{j-1}(n), x_{j+1}(n), \ldots, x_k(n), n+t) : n \in \mathbb{Z}\}$$

is a translated 1-set of class h with respect to the primes $p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_k$. On the other hand, if R_0 is a residue class modulo q, where $(q, p_1 \ldots p_k) = 1$, then the set $\{(\mathbf{x}(n), n+t) : n \in R_0\}$ is a translated q-set with respect to the primes $p_1, \ldots, p_k$.

Suppose $B = I_1 \times \ldots \times I_k \times I^*$, where $I_j \subset U_0$ $(1 \leqslant j \leqslant k)$ and I^* is of the type $[0, Y)$, where Y is positive but otherwise unrestricted. For any $\Omega^{(q)}(t)$ with respect to the primes $p_1, \ldots, p_k$, we define $E[\Omega^{(q)}(t); B]$ by

$$E[\Omega^{(q)}(t); B] = Z[\Omega^{(q)}(t); B] - q^{-1} V(B), \tag{7}$$

where $Z[\Omega^{(q)}(t); B]$ is the number of points of $\Omega^{(q)}(t)$ falling into B, and $V(B)$ is the volume of B. Note that $q^{-1}V(B)$ is the "expected number" of points of $\Omega^{(q)}(t)$ falling into B. The analogues of (7) in lower-dimensional cases are defined in the same way. It is clear that if $B = B_1 \cup B_2$, where $B_1 \cap B_2 = \varnothing$, then

$$E[\Omega^{(q)}(t); B] = E[\Omega^{(q)}(t); B_1] + E[\Omega^{(q)}(t); B_2]. \tag{8}$$

Definition. Let $0 \leqslant s \leqslant h$. Suppose further that, for each $j = 1, \ldots, k$, $I_j = [0, \eta_j)$, where $0 < \eta_j \leqslant 1$ and η_j is an integer multiple of p_j^{-s}. Then we say that B is a box of class s with respect to the primes $p_1, \ldots, p_k$.

We reserve the symbol B^* for such boxes.

LEMMA 3. *Suppose $\Omega^{(q)}$ is a q-set of class h with respect to the primes $p_1, \ldots, p_k$, where $(q, p_1 \ldots p_k) = 1$; and suppose B^* is a box of class $\bar{s}$ with respect to the same primes, where $0 \leqslant s \leqslant h$. Then the function $E[\Omega^{(q)}(t); B^*]$ is periodic in t with period $q(p_1 \ldots p_k)^s$.*

Proof. Since B^* is a box of class s with respect to the primes $p_1, \ldots, p_k$, B^* is a finite and mutually disjoint union of boxes of the type $B = J_1 \times \ldots \times J_k \times I^*$, where, for $j = 1, \ldots, k$, J_j is an elementary p_j-type interval of order s. By (8), it suffices to prove Lemma 3 with B^* replaced by such a box B. Consider, for any $j = 1, \ldots, k$, the set $Q_j = \{n \in Q : x_j(n) \in J_j\}$. Then $Q_j = Q \cap R_j$, where R_j is the associated residue class (modulo p_j^s) of J_j. By the Chinese Remainder Theorem, the set $R_1 \cap \ldots \cap R_k$, $= R$ say, is a residue class modulo $(p_1 \ldots p_k)^s$. Since $(q, p_1 \ldots p_k) = 1$, we see that $Q \cap R$ is a residue class modulo $q(p_1 \ldots p_k)^s$. It follows from (1), (6) and (7) that for the above B,

$$E[\Omega^{(q)}(t); B] = F[t + (Q \cap R); I^*].$$

The proof of Lemma 3 is now complete, since $F[t + (Q \cap R); I^*]$ is clearly periodic with the desired period.

§4. *Statement of the basic result and an important identity.* In this section and the next two, we establish the following result from which Theorem 2 will be deduced in §7. We also assume that W is an even positive integer. Let

$$M = p_1 \ldots p_k; \tag{9}$$

and, for $j = 1, \ldots, k$, write

$$M_j = p_1 \ldots p_{j-1} p_{j+1} \ldots p_k = M p_j^{-1}. \tag{10}$$

MAIN LEMMA. *For a suitable constant $C(p_1, \ldots, p_k, W)$, depending only on W and the primes $p_1, \ldots, p_k$, we have, for any 1-set $\Omega^{(1)}$ of class h with respect to the primes $p_1, \ldots, p_k$, and, for any box B^* of class h with respect to the same primes,*

$$\int_0^{M^h} E[\Omega^{(1)}(t); B^*]^W dt \leqslant C(p_1, \ldots, p_k, W) M^h h^{\frac{1}{2}kW}. \tag{11}$$

The Main Lemma is in fact equivalent to the following superficially more general form.

ALTERNATIVE FORM OF THE MAIN LEMMA. *For the same constant* $C(p_1, \ldots, p_k, W)$ *as in the Main Lemma, for any q-set* $\Omega^{(q)}$ *of class h with respect to the primes* $p_1, \ldots, p_k$, *where* $(q, p_1 \ldots p_k) = 1$, *and for any box* B^* *of class h with respect to the same primes, we have*

$$\int_0^{qM^h} E[\Omega^{(q)}(t) ; B^*]^W dt \leqslant C(p_1, \ldots, p_k, W) q M^h h^{\frac{1}{2}kW}. \tag{12}$$

To see this equivalence, we first note that if $\Omega^{(q)}(t) = \Omega(Q, \mathscr{X} ; t)$, where Q is the residue class n_0 modulo q, then, by Lemma 3, the left-hand side of (12) is equal to

$$q \int_0^{M^h} E[\Omega(Q, \mathscr{X} ; qt' - n_0) ; B^*]^W dt',$$

where $\Omega(Q, \mathscr{X} ; qt' - n_0) = \{(\mathbf{x}(n), n - n_0 + qt') : n \in Q\}$. We now define $\Omega(\mathbb{Z}, \mathscr{X}' ; t') = \{(\mathbf{x}'(n'), n' + t') : n' \in \mathbb{Z}\}$ by writing, for each $n' \in \mathbb{Z}$, $\mathbf{x}'(n') = \mathbf{x}(n_0 + qn')$, so that

$$\{(\mathbf{x}(n), q^{-1}(n - n_0) + t') : n \in Q\} = \{(\mathbf{x}'(n'), n' + t') : n' \in \mathbb{Z}\}. \tag{13}$$

Let $(B')^*$ be derived from $B^* = I_1 \times \ldots \times I_k \times [0, Y)$ by writing $(B')^* = I_1 \times \ldots \times I_k \times [0, q^{-1} Y)$. It now suffices to show that $\Omega(\mathbb{Z}, \mathscr{X}' ; t')$ is a translated 1-set of class h with respect to the primes $p_1, \ldots, p_k$, so that by (7) and (13),

$$\begin{aligned} E[\Omega(Q, \mathscr{X} ; qt' - n_0) ; B^*] &= Z[\Omega(Q, \mathscr{X} ; qt' - n_0) ; B^*] - q^{-1} V(B^*) \\ &= Z[\Omega(\mathbb{Z}, \mathscr{X}' ; t') ; (B')^*] - V((B')^*) \\ &= E[\Omega(\mathbb{Z}, \mathscr{X}' ; t') ; (B')^*], \end{aligned}$$

and our argument is complete. It now remains to show that, for any $j = 1, \ldots, k$, the set $\mathscr{X}' = \{\mathbf{x}'(n') : n' \in \mathbb{Z}\}$ sets up an association of order h with respect to the prime p_j. In other words, we have to show that for any $j = 1, \ldots, k$ and for any s satisfying $0 \leqslant s \leqslant h$, the set of those integers n' for which $x'_j(n')$ falls into any fixed elementary p_j-type interval J of order s constitutes a residue class modulo p_j^s. Since $\Omega(Q, \mathscr{X})$ is a q-set of class h with respect to the primes $p_1, \ldots, p_k$, the set of those $n \in Q$ for which $x_j(n)$ falls into J is contained in a residue class R modulo p_j^s. Then it is clear that

$$\begin{aligned} \{n' : x'_j(n') \in J\} &= \{n' : n_0 + qn' \in Q \wedge x_j(n_0 + qn') \in J\} \\ &= \{n' : n_0 + qn' \in Q \cap R\}, \end{aligned}$$

where $Q \cap R$ is a residue class modulo qp_j^s. Furthermore,

$$n_0 + qn' \in Q \cap R \iff n' \in R',$$

where R' is some residue class modulo p_j^s.

To prove the Main Lemma, we induce on h. For $k \geqslant 2$, we also make use of the corresponding results in a lower-dimensional case. We therefore begin by investigating some properties of $\Omega^{(1)}$ and B^*.

Let $\mathscr{X} = \{\mathbf{x}(n) : n \in \mathbb{Z}\}$ be any arbitrary set such that the set $\Omega = \Omega^{(1)}$ defined by (5) is a 1-set of class h with respect to the primes $p_1, \ldots, p_k$. We keep this set $\mathscr{X}$ fixed throughout.

Suppose

$$B^* = B^*(\boldsymbol{\eta}, Y) = [0, \eta_1) \times \ldots \times [0, \eta_k) \times [0, Y) \tag{14}$$

is a box of class h with respect to the primes $p_1, \ldots, p_k$. For $j = 1, \ldots, k$ and $s = 0, \ldots, h$, let $\xi_{j,s}$ denote the greatest integer multiple of p_j^{-s} not exceeding η_j; furthermore, for $s \neq 0$, write

$$v_{j,s} = p_j^s(\xi_{j,s} - \xi_{j,s-1}).$$

Then $v_{j,s}$ is an integer and $0 \leqslant v_{j,s} < p_j$ for $j = 1, \ldots, k$ and $s = 1, \ldots, h$. Let

$$I^* = [0, Y). \tag{15}$$

For $s = 0, \ldots, h$, let

$$B_s^* = [0, \xi_{1,s}) \times \ldots \times [0, \xi_{k,s}) \times I^*, \tag{16}$$

so that B_s^* is the largest box of class s with respect to the primes $p_1, \ldots, p_k$ which is contained in B^*. We now consider the complement of B_{s-1}^* in B_s^*. We let $B_{1,s}$ denote the part of this which is contained in $[\xi_{1,s-1}, \xi_{1,s}) \times [0, 1)^{k-1} \times I^*$, $B_{2,s}$ denote the part of the remainder which is contained in $[0, 1) \times [\xi_{2,s-1}, \xi_{2,s}) \times [0, 1)^{k-2} \times I^*$, and so on. In other words, for $j = 1, \ldots, k$ and $s = 1, \ldots, h$, we let

$$\begin{aligned} B_{j,s} = {} & [0, \xi_{1,s-1}) \times \ldots \times [0, \xi_{j-1,s-1}) \times [\xi_{j,s-1}, \xi_{j,s}) \\ & \times [0, \xi_{j+1,s}) \times \ldots \times [0, \xi_{k,s}) \times I^*. \end{aligned} \tag{17}$$

Then, it follows that, for $s = 1, \ldots, h$,

$$B_s^* = B_{s-1}^* \cup B_{1,s} \cup \ldots \cup B_{k,s}, \tag{18}$$

and that the union is pairwise disjoint; so it follows from (8) that, for $s = 1, \ldots, h$,

$$E[\Omega(t); B_s^*] = E[\Omega(t); B_{s-1}^*] + \sum_{j=1}^{k} E[\Omega(t); B_{j,s}]. \tag{19}$$

If, for some j, s with $1 \leqslant j \leqslant k$ and $1 \leqslant s \leqslant h$, $B_{j,s} = \varnothing$, then obviously $E[\Omega(t); B_{j,s}] = 0$, and we may omit reference to $E[\Omega(t); B_{j,s}]$ in (19). So we may suppose that all $B_{j,s}$ are non-empty. Then the interval $[\xi_{j,s-1}, \xi_{j,s})$ in (17) is a union of exactly $v_{j,s}$ disjoint elementary p_j-type intervals of order s. We denote them by $J_{j,s,\alpha}$ $(a = 1, \ldots, v_{j,s})$, and, for future reference, we suppose they correspond respectively to the residue classes $R(m(j,s,\alpha),p_j^s)$. For $\alpha = 1, \ldots, v_{j,s}$, we write

$$B_{j,s,\alpha} = [0,\xi_{1,s-1})\times\ldots\times[0,\xi_{j-1,s-1})\times J_{j,s,\alpha}$$
$$\times[0,\xi_{j+1,s})\times\ldots\times[0,\xi_{k,s})\times I^*. \tag{20}$$

Then we see that, for $j = 1, \ldots, k$ and $s = 1, \ldots, h$,

$$B_{j,s} = \bigcup_{\alpha=1}^{v_{j,s}} B_{j,s,\alpha},$$

and the union is pairwise disjoint, so that by (8),

$$E[\Omega(t);B_{j,s}] = \sum_{\alpha=1}^{v_{j,s}} E[\Omega(t);B_{j,s,\alpha}]. \tag{21}$$

For $j = 1, \ldots, k$ and $s = 1, \ldots, h-1$, let

$$\bar{J}_{j,s} = [\xi_{j,s}, \xi_{j,s}+p_j^{-s}), \tag{22}$$

so that $\bar{J}_{j,s}$ is the elementary p_j-type interval of order s containing the complement of $[0,\xi_{j,s})$ in $[0,\eta_j)$. We modify $B_{j,h}$ (see (17) with $s = h$) by replacing $[\xi_{j,h-1},\xi_{j,h})$ by $\bar{J}_{j,s}$, which is the elementary p_j-type interval of order s containing $[\xi_{j,h-1},\xi_{j,h})$. Accordingly, for $j = 1, \ldots, k$ and $s = 1, \ldots, h-1$, we let the modification $\bar{B}_{j,s}$ be defined by

$$\bar{B}_{j,s} = [0,\xi_{1,h-1})\times\ldots\times[0,\xi_{j-1,h-1})\times\bar{J}_{j,s}$$
$$\times[0,\xi_{j+1,h})\times\ldots\times[0,\xi_{k,h})\times I^*. \tag{23}$$

Note that $B_{j,h}$ has $h-1$ modifications, namely $\bar{B}_{j,1}, \ldots, \bar{B}_{j,h-1}$.

Our next lemma contains the key idea underlying Roth's proof of his Basic Lemma in [**6**].

LEMMA 4. *For $j = 1, \ldots, k$ and for $\alpha = 1, \ldots, v_{j,h}$, we have*

$$\sum_{a=0}^{p_j-1} E[\Omega(t+aM_j^h p_j^{h-1});B_{j,h,\alpha}] = E[\Omega(t);\bar{B}_{j,h-1}]. \tag{24}$$

Also, for $j = 1, \ldots, k$ and $s = 2, \ldots, h-1$, we have

$$\sum_{a=0}^{p_j-1} E[\Omega(t+aM_j^h p_j^{s-1});\bar{B}_{j,s}] = E[\Omega(t);\bar{B}_{j,s-1}]. \tag{25}$$

Proof. We first prove (24). Recall the definitions of $B_{j,h,\alpha}$ (see (20) with $s = h$) and $\bar{B}_{j,h-1}$ (see (23) with $s = h-1$). We note that the only difference is that $J_{j,h,\alpha}$ in the former is replaced by $\bar{J}_{j,h-1}$ in the latter. By (22), we see that $\bar{J}_{j,h-1}$ is the elementary p_j-type interval of order $h-1$ containing $J_{j,h,\alpha}$. Hence, by (4), the associated residue class $R(m(j,h,\alpha),p_j^h)$ of $J_{j,h,\alpha}$ is contained in the associated residue class, R_0 say, of $\bar{J}_{j,h-1}$. So R_0 is the residue class modulo p_j^{h-1} containing $R(m(j,h,\alpha), p_j^h)$. Let J_i, $i = 1, \ldots, j-1, j+1, \ldots, k$, denote any elementary p_i-type interval of order h. Then we see that, since $[0, \xi_{i,h-1})$ and $[0, \xi_{i,h})$ are the union of a finite number of mutually disjoint intervals of the type J_i, it follows from (8) that to prove (24), it suffices to prove that

$$\sum_{a=0}^{p_j-1} E[\Omega(t+aM_j^h p_j^{h-1}); J_1 \times \ldots \times J_{j-1} \times J_{j,h,\alpha} \times J_{j+1} \times \ldots \times J_k \times I^*]$$

$$= E[\Omega(t); J_1 \times \ldots \times J_{j-1} \times \bar{J}_{j,h-1} \times J_{j+1} \times \ldots \times J_k \times I^*]. \tag{26}$$

For $i = 1, \ldots, j-1, j+1, \ldots, k$, let R_i denote the associated residue class of J_i. Then the summand on the left-hand side of (26) is, by (1) and (7), equal to

$$F\left[t+aM_j^h p_j^{h-1}+\left(R_1 \cap \ldots \cap R_{j-1} \cap R(m(j,h,\alpha), p_j^h) \cap R_{j+1} \cap \ldots \cap R_k\right); I^*\right]$$

$$= F\left[t+\left(R_1 \cap \ldots \cap R_{j-1} \cap \left(aM_j^h p_j^{h-1}+R(m(j,h,\alpha), p_j^h)\right) \cap R_{j+1} \cap \ldots \cap R_k\right); I^*\right]. \tag{27}$$

On the other hand,

$$\bigcup_{a=0}^{p_j-1} \left(aM_j^h p_j^{h-1}+R(m(j,h,\alpha), p_j^h)\right) = R_0. \tag{28}$$

It follows from (27) and (28) that the left-hand side of (26), and so also the right-hand side, is equal to

$$F[t+(R_1 \cap \ldots \cap R_{j-1} \cap R_0 \cap R_{j+1} \cap \ldots \cap R_k); I^*].$$

This completes the proof of (24). Equation (25) follows similarly, for if R_s and R_{s-1} are respectively the associated residue classes of $\bar{J}_{j,s}$ and $\bar{J}_{j,s-1}$, then the analogue of (28) is

$$\bigcup_{a=0}^{p_j-1} (aM_j^h p_j^{s-1}+R_s) = R_{s-1}.$$

This completes the proof of Lemma 4.

For $s = 1, \ldots, h$, write, for the sake of simplicity,

$$G_s(t) = \sum_{j=1}^{k} E[\Omega(t); B_{j,s}]. \tag{29}$$

Let

$$T_0 = \int_0^{M^h} E[\Omega(t); B_{h-1}^*]^W dt; \tag{30}$$

and, for $w = 2, \dots, W$,

$$T_w = \int_0^{M^h} E[\Omega(t); B^*_{h-1}]^{W-w} G^w_h(t)dt. \tag{31}$$

For $j = 1, \dots, k$, we write

$$S_j = \int_0^{M^h} E[\Omega(t); B^*_0]^{W-1} E[\Omega(t); \bar{B}_{j,1}]dt; \tag{32}$$

further, for $s = 1, \dots, h-1$ and $w = 1, \dots, W-1$, we write

$$S_{j,s,w} = \int_0^{M^h} E[\Omega(t); B^*_{s-1}]^{W-w-1} G^w_s(t) E[\Omega(t); \bar{B}_{j,s}]dt. \tag{33}$$

Our proof of the Main Lemma is based on the following identity.

LEMMA 5. *We have, for $h \geqslant 2$, that*

$$\int_0^{M^h} E[\Omega(t); B^*]^W dt = T_0 + \sum_{w=2}^{W} \binom{W}{w} T_w + W \sum_{j=1}^{k} v_{j,h} p_j^{-(h-1)} S_j$$
$$+ W \sum_{j=1}^{k} v_{j,h} \sum_{s=1}^{h-1} p_j^{-(h-s)} \sum_{w=1}^{W-1} \binom{W-1}{w} S_{j,s,w}. \tag{34}$$

Proof. We note, by (14), (15) and (16), that $B^* = B^*_h$; so, by taking W-th powers on both sides of (19) with $s = h$, and using binomial expansion on the right-hand side, we have, in view of (29),

$$E[\Omega(t); B^*]^W = E[\Omega(t); B^*_{h-1}]^W + W \sum_{j=1}^{k} E[\Omega(t); B^*_{h-1}]^{W-1} E[\Omega(t); B_{j,h}]$$
$$+ \sum_{w=2}^{W} \binom{W}{w} E[\Omega(t); B^*_{h-1}]^{W-w} G^w_h(t).$$

By (30) and (31), we see that to prove Lemma 5, it remains to show that, for $j = 1, \dots, k$,

$$\int_0^{M^h} E[\Omega(t); B^*_{h-1}]^{W-1} E[\Omega(t); B_{j,h}]dt$$
$$= v_{j,h} p_j^{-(h-1)} S_j + v_{j,h} \sum_{s=1}^{h-1} p_j^{-(h-s)} \sum_{w=1}^{W-1} \binom{W-1}{w} S_{j,s,w}. \tag{35}$$

Now since B^*_{h-1} is a box of class $h-1$ with respect to the primes $p_1, \ldots, p_k$, we know, by Lemma 3, that $E[\Omega(t); B^*_{h-1}]$ is periodic in t with period M^{h-1}, and therefore also periodic with period $M_j^h p_j^{h-1}$, for every $j = 1, \ldots, k$. So, by (21) with $s = h$ and (24), we have, for $j = 1, \ldots, k$ and writing V_j for the left-hand side of (35),

$$V_j = \sum_{\alpha=1}^{v_{j,h}} \int_0^{M^h} E[\Omega(t); B^*_{h-1}]^{W-1} E[\Omega(t); B_{j,h,\alpha}]\,dt$$

$$= v_{j,h} p_j^{-1} \int_0^{M^h} E[\Omega(t); B^*_{h-1}]^{W-1} E[\Omega(t); \bar{B}_{j,h-1}]\,dt. \tag{36}$$

Now, by (19) and (29), we have, for $s = 1, \ldots, h-1$,

$$E[\Omega(t); B^*_s]^{W-1} = E[\Omega(t); B^*_{s-1}]^{W-1} + \sum_{w=1}^{W-1} \binom{W-1}{w} E[\Omega(t); B^*_{s-1}]^{W-w-1} G_s^w(t). \tag{37}$$

On combining (36) and the case $s = h-1$ of (37) and (33), we have, for $j = 1, \ldots, k$,

$$v_{j,h}^{-1} V_j = p_j^{-1} \int_0^{M^h} E[\Omega(t); B^*_{h-2}]^{W-1} E[\Omega(t); \bar{B}_{j,h-1}]\,dt + p_j^{-1} \sum_{w=1}^{W-1} \binom{W-1}{w} S_{j,h-1,w}. \tag{38}$$

For $s = 1, \ldots, h-2$, we know that B^*_s is a box of class s with respect to the primes $p_1, \ldots, p_k$, and so, by Lemma 3, (25), (37) and (33), we have, for $j = 1, \ldots, k$ and $s = 1, \ldots, h-2$,

$$\int_0^{M^h} E[\Omega(t); B^*_s]^{W-1} E[\Omega(t); \bar{B}_{j,s+1}]\,dt$$

$$= p_j^{-1} \int_0^{M^h} E[\Omega(t); B^*_s]^{W-1} E[\Omega(t); \bar{B}_{j,s}]\,dt$$

$$= p_j^{-1} \int_0^{M^h} E[\Omega(t); B^*_{s-1}]^{W-1} E[\Omega(t); \bar{B}_{j,s}]\,dt + p_j^{-1} \sum_{w=1}^{W-1} \binom{W-1}{w} S_{j,s,w}. \tag{39}$$

In view of (32) the result (35) now follows from (38) by repeated application of (39). This completes the proof of Lemma 5.

§5. *The case* $k = 1$. The following lemma is only applicable for the case $k = 1$.

LEMMA 6. *For* $s = 1, \ldots, h$,

$$|G_s(t)| = |E[\Omega(t); B_{1,s}]| < p_1; \tag{40}$$

also, for $s = 1, \ldots, h-1$,

$$|E[\Omega(t); \bar{B}_{1,s}]| \leqslant 1. \tag{41}$$

Proof. Recall (21) with $j = 1$. For $s = 1, \ldots, h$,

$$E[\Omega(t); B_{1,s}] = \sum_{\alpha=1}^{v_{1,s}} E[\Omega(t); B_{1,s,\alpha}].$$

Since $v_{1,s} < p_1$, we see that to prove (40) it suffices to show that, for each $\alpha = 1, \ldots, v_{1,s}$,

$$|E[\Omega(t); B_{1,s,\alpha}]| \leqslant 1. \tag{42}$$

Now on recalling (20), we know, for $s = 1, \ldots, h$ and $\alpha = 1, \ldots, v_{1,s}$, that $B_{1,s,\alpha} = J_{1,s,\alpha} \times I^*$, where $J_{1,s,\alpha}$ is an elementary p_1-type interval of order s. It follows from (1), (7) and the remark before (20) that

$$E[\Omega(t); B_{1,s,\alpha}] = F\big[t + R\big(m(1, s, \alpha), p_1^s\big); I^*\big]. \tag{43}$$

(42) now follows from (43) and (2). Inequality (41) follows similarly, since, for $s = 1, \ldots, h-1$, we know, by (23) and (22), that $\bar{B}_{1,s} = \bar{J}_{1,s} \times I^*$, and $\bar{J}_{1,s}$ is an elementary p_1-type interval of order s. This completes the proof of Lemma 6.

We shall prove by induction on h that the Main Lemma for $k = 1$ holds for a constant C satisfying

$$C = C(p_1, W) = (2^{W+1} p_1)^W. \tag{44}$$

Note that, in particular,

$$C > p_1^W. \tag{45}$$

Suppose $B^* = I_1 \times I^*$, where $I_1 = [0, 1)$. Then by (1), (7) and (2),

$$|E[\Omega(t); B^*]| = |F[t + \mathbb{Z}; I^*]| \leqslant 1,$$

and the Main Lemma for $k = 1$ is proved. We may therefore suppose that $I_1 \neq [0, 1)$, and hence, by (16), that $B_0^* = \varnothing$. For $h = 1$, we know, by (18) and (40), that

$$|E[\Omega(t); B^*]| = |E[\Omega(t); B_1^*]| = |E[\Omega(t); B_{1,1}]| < p_1.$$

So in view of (45), the Main Lemma for $k = 1$ holds for $h = 1$. Suppose now that $h > 1$ and that the Main Lemma for $k = 1$ holds, when h is replaced by any smaller positive integer, so that in particular, for any $s = 2, \dots, h$, we have, in view of Lemma 3,

$$\int_0^{M^h} E[\Omega(t)\,;B^*_{s-1}]^W dt \leqslant CM^h(s-1)^{\frac{1}{2}W}. \tag{46}$$

Applying Hölder's inequality, (40) and (46) to (31), we have, for $w = 2, \dots, W$,

$$T_w \leqslant C^{(W-w)/W} p_1^w M^h (h-1)^{\frac{1}{2}(W-w)} < C^{1-1/W} p_1 M^h (h-1)^{\frac{1}{2}W-1}$$

in view of (45), so that

$$\sum_{w=2}^{W} \binom{W}{w} T_w \leqslant C^{1-1/W} 2^W p_1 M^h (h-1)^{\frac{1}{2}W-1}. \tag{47}$$

On the other hand, since $B_0^* = \varnothing$, we have $E[\Omega(t)\,;B_0^*] = 0$, and so, by (32),

$$S_1 = 0. \tag{48}$$

Applying Hölder's inequality, (40), (41) and (46) to (33), we have, for $s = 2, \dots, h-1$ and $w = 1, \dots, W-1$, in view of (45),

$$S_{1,s,w} \leqslant C^{(W-w-1)/W} p_1^w M^h (s-1)^{\frac{1}{2}(W-w-1)} < C^{1-1/W} M^h (h-1)^{\frac{1}{2}W-1}.$$

Furthermore

$$S_{1,1,W-1} \leqslant p_1^{W-1} M^h < C^{1-1/W} M^h (h-1)^{\frac{1}{2}W-1}.$$

For $w = 1, \dots, W-2$, we have $S_{1,1,w} = 0$. It follows that

$$W v_{1,h} \sum_{s=1}^{h-1} p_1^{-(h-s)} \sum_{w=1}^{W-1} \binom{W-1}{w} S_{1,s,w} \leqslant C^{1-1/W} 2^{W-1} W p_1 M^h (h-1)^{\frac{1}{2}W-1}. \tag{49}$$

On the other hand, by (30) and (46),

$$T_0 \leqslant CM^h (h-1)^{\frac{1}{2}W}. \tag{50}$$

Combining (34), (47), (48), (49) and (50),

$$\begin{aligned}
\int_0^{M^h} E[\Omega(t)\,;B^*]^W dt &\leqslant CM^h(h-1)^{\frac{1}{2}W} + C^{1-1/W}(2^W + 2^{W-1}W) p_1 M^h (h-1)^{\frac{1}{2}W-1} \\
&\leqslant CM^h\{(h-1)^{\frac{1}{2}W} + C^{-1/W} 2^W W p_1 (h-1)^{\frac{1}{2}W-1}\} \\
&= CM^h\{(h-1)^{\frac{1}{2}W} + \tfrac{1}{2}W(h-1)^{\frac{1}{2}W-1}\} \\
&\leqslant CM^h h^{\frac{1}{2}W},
\end{aligned}$$

in view of (44). The proof of the Main Lemma for $k = 1$ is now complete.

§6. *The case $k \geqslant 2$.* We suppose, for every $j = 1, \dots, k$, that the corresponding Main Lemma for q-sets and boxes with respect to the primes $p_1, \dots, p_{j-1}, p_{j+1}, \dots, p_k$ holds with $C_j(p_1, \dots, p_{j-1}\ p_{j+1}, \dots, p_k, W)$ and M_j in place of $C(p_1, \dots, p_k, W)$ and M respectively, and with the exponent of h replaced by $\frac{1}{2}(k-1)W$. We let $C_0 = C_0(p_1, \dots, p_k, W)$ be defined by

$$C_0 = 1 + \max_{1 \leqslant j \leqslant k} C_j(p_1, \dots, p_{j-1}, p_{j+1}, \dots, p_k, W). \tag{51}$$

We shall prove, by induction on h, that the Main Lemma holds for a constant C satisfying

$$C = C(p_1, \dots, p_k, W) = 2^{kW^2+W}(p_1 \dots p_k)^{W+1} C_0(p_1, \dots, p_k, W). \tag{52}$$

Clearly

$$C > (p_1 \dots p_k)^{W+1} C_0 \tag{53}$$

and

$$C > (p_1 \dots p_k)^W. \tag{54}$$

First of all, we must establish a link which enables us to use the analogues of the Main Lemma in lower-dimensional cases.

For $j = 1, \dots, k$, $s = 1, \dots, h$ and $\alpha = 1, \dots, v_{j,s}$, we define the set $\Omega_{j,s,\alpha}(t)$ of points in k-dimensional space by

$$\Omega_{j,s,\alpha}(t) = \{(x_1(n), \dots, x_{j-1}(n), x_{j+1}(n), \dots, x_k(n), n+t : n \in R(m(j, s, \alpha), p_j^s)\},$$

where $R(m(j, s, \alpha), p_j^s)$ is the associated residue class modulo p_j^s of $J_{j,s,\alpha}$. Then, for every real number t, we know, by Lemma 2, that $\Omega_{j,s,\alpha}(t)$ is a translated p_j^s-set of class h with respect to the primes $p_1, \dots, p_{j-1}, p_{j+1}, \dots, p_k$. On the other hand, let $B_{j,s}^*$ be defined for the above values of j and s by

$$B_{j,s}^* = [0, \xi_{1,s-1}) \times \dots \times [0, \xi_{j-1,s-1}) \times [0, \xi_{j+1,s}) \times \dots \times [0, \xi_{k,s}) \times I^*.$$

Then $B_{j,s}^*$ is a box of class s with respect to the primes $p_1, \dots, p_{j-1}, p_{j+1}, \dots, p_k$. It is easy to see that, for $j = 1, \dots, k$, $s = 1, \dots, h$ and $\alpha = 1, \dots, v_{j,s}$,

$$E[\Omega(t); B_{j,s,\alpha}] = E[\Omega_{j,s,\alpha}(t); B_{j,s}^*]. \tag{55}$$

It follows, on our assumption of the lower-dimensional cases of the Main Lemma, and hence its alternative form, in view of (55), (51), Lemma 3, (9) and (10), that for every $j = 1, \dots, k$, $s = 1, \dots, h$ and $\alpha = 1, \dots, v_{j,s}$,

$$\int_0^{M^h} E[\Omega(t); B_{j,s,\alpha}]^W dt = M^{h-s} \int_0^{p_j^s M_j^s} E[\Omega_{j,s,\alpha}(t); B_{j,s}^*]^W dt$$

$$\leqslant C_j M^h s^{\frac{1}{2}(k-1)W} < C_0 M^h s^{\frac{1}{2}(k-1)W}. \tag{56}$$

Similarly, for $j = 1, \ldots, k$ and $s = 1, \ldots, h-1$, on relating $E[\Omega(t); \bar{B}_{j,s}]$ to the discrepancy of certain translated p_j^s-set with respect to the primes $p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_k$ in the box $B_{j,h}^*$, we have

$$\int_0^{M^h} E[\Omega(t); \bar{B}_{j,s}]^W dt < C_0 M^h h^{\frac{1}{2}(k-1)W}. \tag{57}$$

We know, by (29) and (21), that, for $s = 1, \ldots, h$,

$$\begin{aligned}\int_0^{M^h} G_s^W(t)dt &= \int_0^{M^h} \left(\sum_{j=1}^{k} \sum_{\alpha=1}^{v_{j,s}} E[\Omega(t); B_{j,s,\alpha}] \right)^W dt \\ &\leqslant \left(\sum_{j=1}^{k} v_{j,s} \right)^W \sum_{j=1}^{k} \sum_{\alpha=1}^{v_{j,s}} \int_0^{M^h} E[\Omega(t); B_{j,s,\alpha}]^W dt; \end{aligned} \tag{58}$$

so that, on combining (56) and (58), we have, for $s = 1, \ldots, h$,

$$\begin{aligned}\int_0^{M^h} G_s^W(t)dt &\leqslant (p_1 + \ldots + p_k)^{W+1} C_0 M^h s^{\frac{1}{2}(k-1)W} \\ &< (p_1 \ldots p_k)^{W+1} C_0 M^h s^{\frac{1}{2}(k-1)W}. \end{aligned} \tag{59}$$

We may assume that $B^* = I_1 \times \ldots \times I_k \times I^*$, where for some $j = 1, \ldots, k$, $I_j \neq [0, 1)$. Otherwise, by (1), (7) and (2), we know that $|E[\Omega(t); B^*]| = |F[t + \mathbb{Z}; I^*]| \leqslant 1$, and the Main Lemma follows. By (16), we see that $B_0^* = \varnothing$. On the other hand, it is easy to see, from the definition of $\Omega(t)$, that for any fixed t and $\boldsymbol{\eta}$, $E[\Omega(t); B_1^*(\boldsymbol{\eta}, Y)]$ is periodic in Y with period $p_1 \ldots p_k$, so that we may assume that $0 \leqslant Y < p_1 \ldots p_k$ when estimating $E[\Omega(t); B_1^*]$. With this restriction on Y, $Z[\Omega(t); B_1^*] \leqslant p_1 \ldots p_k$ and $V(B_1^*) < p_1 \ldots p_k$, so, by (7), $|E[\Omega(t); B_1^*]| \leqslant p_1 \ldots p_k$. Hence, by (54), we see that the Main Lemma holds for $h = 1$ with C defined by (52).

Suppose now that $h \geqslant 2$, and that the Main Lemma holds when h is replaced by any smaller positive integer, so that, in particular, for $s = 2, \ldots, h$, in view of Lemma 3,

$$\int_0^{M^h} E[\Omega(t); B_{s-1}^*]^W dt \leqslant C M^h (s-1)^{\frac{1}{2}kW}. \tag{60}$$

Applying Hölder's inequality, (59) and (60) to (31), for $w = 2, \ldots, W$, we obtain

$$\begin{aligned} T_w &\leqslant C^{(W-w)/W} C_0^{w/W} (p_1 \ldots p_k)^{w(W+1)/W} M^h (h-1)^{\frac{1}{2}k(W-w)} h^{\frac{1}{2}(k-1)w} \\ &\leqslant C^{1-1/W} C_0^{1/W} (p_1 \ldots p_k)^{(W+1)/W} 2^{(k-1)W} M^h (h-1)^{\frac{1}{2}kW-1} \end{aligned}$$

by (53). Hence

$$\sum_{w=2}^{W}\binom{W}{w}T_w \leqslant C^{1-1/W}C_0^{1/W}(p_1 \dots p_k)^{(W+1)/W}2^{kW}M^h(h-1)^{\frac{1}{2}kW-1}. \tag{61}$$

On the other hand, since $B_0^* = \varnothing$, we know that $E[\Omega(t); B_0^*] = 0$, and so by (32), we deduce that for every $j = 1, \dots, k$,

$$S_j = 0. \tag{62}$$

Applying Hölder's inequality, (57), (59) and (60) to (33), for $j = 1, \dots, k$, $s = 2, \dots, h-1$ and $w = 1, \dots, W-1$, in view of (53), we obtain

$$\begin{aligned} S_{j,s,w} &\leqslant C^{(W-w-1)/W}C_0^{(w+1)/W}(p_1 \dots p_k)^{w(W+1)/W}\, M^h(s-1)^{\frac{1}{2}k(W-w-1)}s^{\frac{1}{2}(k-1)w}h^{\frac{1}{2}(k-1)} \\ &\leqslant C^{1-1/W}C_0^{1/W}2^{\frac{1}{2}(k-1)}M^h(h-1)^{\frac{1}{2}kW-1}. \end{aligned}$$

Furthermore,

$$\begin{aligned} S_{j,1,W-1} &\leqslant C_0(p_1 \dots p_k)^{(W-1)(W+1)/W}M^h h^{\frac{1}{2}(k-1)} \\ &\leqslant C^{1-1/W}C_0^{1/W}2^{\frac{1}{2}(k-1)}M^h(h-1)^{\frac{1}{2}kW-1}. \end{aligned}$$

For $j = 1, \dots, k$ and $w = 1, \dots, W-2$, we know that $S_{j,1,w} = 0$. It follows that

$$\begin{aligned} W\sum_{j=1}^{k} v_{j,h}\sum_{s=1}^{h-1} p_j^{-(h-s)}\sum_{w=1}^{W-1}\binom{W-1}{w}S_{j,s,w} \\ &\leqslant C^{1-1/W}C_0^{1/W}(p_1 + \dots + p_k)2^{\frac{1}{2}(k-1)}2^{W-1}WM^h(h-1)^{\frac{1}{2}kW-1} \\ &\leqslant C^{1-1/W}C_0^{1/W}(p_1 \dots p_k)^{(W+1)/W}2^{kW}M^h(h-1)^{\frac{1}{2}kW-1}, \end{aligned} \tag{63}$$

since $2^{\frac{1}{2}(k-1)}2^{W-1}W \leqslant 2^{\frac{1}{2}(k-1)+2W-2} < 2^{kW}$. Also, by (30) and (60), we deduce that

$$T_0 \leqslant CM^h(h-1)^{\frac{1}{2}kW}. \tag{64}$$

Combining (34), (61), (62), (63) and (64), in view of (52), we obtain

$$\begin{aligned} \int_0^{M^h} E[\Omega(t); B^*]^W dt &\leqslant CM^h\{(h-1)^{\frac{1}{2}kW} + 2C^{-1/W}C_0^{1/W}(p_1 \dots p_k)^{(W+1)/W}2^{kW}(h-1)^{\frac{1}{2}kW-1}\} \\ &= CM^h\{(h-1)^{\frac{1}{2}kW} + (h-1)^{\frac{1}{2}kW-1}\} \\ &< CM^h h^{\frac{1}{2}kW}. \end{aligned}$$

This completes the proof of the Main Lemma.

§7. *Proof of Theorem* 2. We now take $p_1, \dots, p_k$ to be the first k primes. For any natural number $N \geqslant 2$, let

$$h = [\log_2 N] + 1 . \tag{65}$$

Then, for every $j = 1, \dots, k$, we have

$$N \leqslant p_j^h . \tag{66}$$

For any $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)$ in U_1^k, and, for any real Y satisfying

$$0 < Y \leqslant N , \tag{67}$$

let $B(\boldsymbol{\theta}, Y)$ be the box defined by

$$B(\boldsymbol{\theta}, Y) = [0, \theta_1) \times \dots \times [0, \theta_k) \times [0, Y) .$$

Let $\boldsymbol{\eta} = \boldsymbol{\eta}(\boldsymbol{\theta}) = (\eta_1, \dots, \eta_k)$ be defined such that, for every j satisfying $1 \leqslant j \leqslant k$,

$$\eta_j = \eta_j(\theta_j) = -p_j^{-h}[-p_j^h \theta_j] , \tag{68}$$

i.e. η_j is the least integer multiple of p_j^{-h} not less than θ_j.

Let $\Omega(t) = \Omega'(t)$, where $\Omega'(t)$ is defined by (3) in terms of the first M^h terms of the Hammersley (Halton) sequence.

LEMMA 7. *For any $\boldsymbol{\theta}$ in U_1^k and any Y satisfying* (67),

$$|E[\Omega(t); B(\boldsymbol{\theta}, Y)] - E[\Omega(t); B^*(\boldsymbol{\eta}(\boldsymbol{\theta}), Y)]| \leqslant k .$$

Proof. For $j = 0, \dots, k$, for any fixed $\boldsymbol{\theta}$ in U_1^k and for any fixed Y satisfying (67), let

$$B^{(j)} = B^{(j)}(\boldsymbol{\theta}, Y) = [0, \eta_1) \times \dots \times [0, \eta_j) \times [0, \theta_{j+1}) \times \dots \times [0, \theta_k) \times [0, Y) .$$

Then clearly $B^{(0)} = B(\boldsymbol{\theta}, Y)$ and $B^{(k)} = B^*(\boldsymbol{\eta}(\boldsymbol{\theta}), Y)$. It is easy to see that to prove Lemma 7, it suffices to prove that, for every $j = 1, \dots, k$,

$$|E[\Omega(t); B^{(j)}] - E[\Omega(t); B^{(j-1)}]| \leqslant 1 .$$

For each $j = 1, \dots, k$, we know, by (68), that $\eta_j \geqslant \theta_j$ and $\eta_j - \theta_j < p_j^{-h}$. It follows from (66) and (67) that

$$0 \leqslant V(B^{(j)}) - V(B^{(j-1)}) < 1 .$$

By (7) with $q = 1$, it remains to show, for $j = 1, \dots, k$, that

$$0 \leqslant Z[\Omega(t); B^{(j)}] - Z[\Omega(t); B^{(j-1)}] \leqslant 1 . \tag{69}$$

The first inequality of (69) is obvious. On the other hand, the difference in question in (69) does not exceed the number of n in $[-t, Y-t)$ for which $X_j(n)$ lies in

$[\eta_j - p_j^{-h}, \eta_j)$. This interval is an elementary p_j-type interval of order h, so that, in view of (66) and (67), and, since $\Omega'(t)$ is of class h, the difference is either 0 or 1. This proves (69), and the proof of Lemma 7 is now complete.

For any even positive integer W, we know, by Lemma 7, that for any $\boldsymbol{\theta}$ in U_1^k and any real Y satisfying $0 < Y \leqslant N$,

$$E[\Omega(t); B(\boldsymbol{\theta}, Y)]^W \leqslant 2^W E[\Omega(t); B^*(\boldsymbol{\eta}(\boldsymbol{\theta}), Y)]^W + (2k)^W. \tag{70}$$

By the Main Lemma, we have

$$\int_0^{M^h} \int_{U_1^k} \int_0^N E[\Omega(t); B^*(\boldsymbol{\eta}(\boldsymbol{\theta}), Y)]^W dt d\theta_1 \dots d\theta_k dY \ll_{k,W} M^h h^{\frac{1}{2}kW} N. \tag{71}$$

Combining (70) and (71), we see that there is a real number t^*, satisfying $0 \leqslant t^* < M^h$, such that

$$\int_{U_1^k} \int_0^N E[\Omega(t^*); B(\boldsymbol{\theta}, Y)]^W d\theta_1 \dots d\theta_k dY \ll_{k,W} (\log N)^{\frac{1}{2}kW} N, \tag{72}$$

in view of (65). Note that the value of t^* may depend on the value of W. On recalling the definition of $\Omega'(t)$ and (6), we see from (72) that the set

$$\mathscr{P} = \{(X_1(n), \dots, X_k(n), N^{-1}(n+t^*)) : 0 \leqslant n+t^* < N\} \tag{73}$$

gives a proof of Theorem 2 for even positive integers W.

The case for general positive W follows immediately.

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10F40: *NUMBER THEORY; Diophantine approximation, Distribution modulo one.*

Received on the 30*th of May*, 1980.