ON IRREGULARITIES OF DISTRIBUTION AND APPROXIMATE EVALUATION OF CERTAIN FUNCTIONS

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§ 1. Introduction

LET $U=[0,1]$, and let $K\geqslant 2$ be an integer. Suppose that g is a Lebesgue-integrable function, not necessarily bounded, in U^K, and that h is any function in U^K. Let $\mathcal{P}=\mathcal{P}(K,N)$ be a distribution of N points in U^K such that $h(\mathbf{y})$ is finite for every $\mathbf{y}\in\mathcal{P}$. For any $\mathbf{x}=(x_1,\ldots,x_K)\in U^K$, let $B(\mathbf{x})$ denote the box consisting of all $\mathbf{y}=(y_1,\ldots,y_K)\in U^K$ satisfying $0\leqslant y_j<x_j$ $(j=1,\ldots,K)$, and write

$$Z[\mathcal{P};h;B(\mathbf{x})]=\sum_{\mathbf{y}\in\mathcal{P}\cap B(\mathbf{x})}h(\mathbf{y}). \tag{1}$$

Let μ denote the Lebesgue measure in U^K, and write

$$D[\mathcal{P};h;g;B(\mathbf{x})]=Z[\mathcal{P};h;B(\mathbf{x})]-N\int_{B(\mathbf{x})}g(\mathbf{y})\,\mathrm{d}\mu. \tag{2}$$

Note that the term

$$N\int_{B(\mathbf{x})}g(\mathbf{y})\,\mathrm{d}\mu$$

can be interpreted as the product of the "expected" number of points of $\mathcal{P}$ in $B(\mathbf{x})$ and the "average" value of g in $B(\mathbf{x})$. It follows that the term $D[\mathcal{P};g;g;B(\mathbf{x})]$ measures the "discrepancy" between a sum of g over points of $\mathcal{P}$ in $B(\mathbf{x})$ and its expected value. In the case when $h=g$ is a non-zero constant function, the problem reduces to one purely concerned with the irregularities of distribution of the set $\mathcal{P}$. For references, see Roth [**9**], Schmidt [**13, 14**] and Halász [**4**].

The purpose of this paper is to use an elaboration of the ideas in Roth [**9**] and Schmidt [**14**] to prove

THEOREM 1. *Suppose that g is a Lebesgue-integrable function in U^K. Suppose further that there exists a measurable subset S of U^K such that*

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$\mu(S)>0$ and $g(\mathbf{y})\neq 0$ for every $\mathbf{y}\in S$. Then, for every real number $W>1$, there exists a positive constant $c_1=c_1(g, W)$ such that for every distribution $\mathscr{P}$ of N points in U^K and for every function h bounded in U^K,

$$\int_{U^K} |D[\mathscr{P}; h; g; B(\mathbf{x})]|^W \, d\mu > c_1(g, W)(\log N)^{\frac{1}{2}(K-1)W}. \tag{3}$$

As an immediate consequence of Theorem 1, we have

COROLLARY 1. *Suppose that g satisfies the hypotheses of Theorem 1. Then there exists a positive constant $c_2=c_2(g)$ such that for every distribution $\mathscr{P}$ of N points in U^K and for every function h bounded in U^K,*

$$\sup_{\mathbf{x}\in U^K} |D[\mathscr{P}; h; g; B(\mathbf{x})]| > c_2(g)(\log N)^{\frac{1}{2}(K-1)}.$$

The case $h=g=1$ in U^K is precisely Theorem 1 of Schmidt [**14**], a generalization of Roth [**9**]. Furthermore, the estimates in this case are essentially sharp (see Davenport [**3**], Roth [**10, 11, 12**] and Chen [**1, 2**]). However, apart from this very special case, very little is known about the behaviour of the function (2). If one combines a result of Halton [**5**] with the generalization by Hlawka [**6**] of Koksma's inequality [**7**], then given any function g of bounded variation on U^K in the sense of Hardy and Krause, and given any natural number $N\geq 2$, there exists a distribution $\mathscr{P}$ of N points in U^K such that

$$D[\mathscr{P}; g; g; B(\mathbf{1})] \ll_g (\log N)^{K-1}.$$

This, however, is again another very restricted special case. The interested reader is referred to Kuipers and Niederreiter [**8**], Chapter 2, Section 5.

As an application of Theorem 1, we shall consider functions in U^K of the following type.

DEFINITION. We denote by $\mathscr{F}(K)$ the class of all functions of the type

$$C + \int_{B(\mathbf{x})} g(\mathbf{y}) \, d\mu$$

in U^K, where C is a real constant, and where g satisfies the hypotheses of Theorem 1.

Theorem 1 can then be applied to show that functions in $\mathscr{F}(K)$ cannot be approximated very well by certain "simple" functions.

DEFINITION. By an M-simple function in U^K, we mean a function ϕ, defined by

$$\phi(\mathbf{x}) = \sum_{i=1}^{M} m_i \chi_{B_i}(\mathbf{x})$$

for all $\mathbf{x} \in U^K$, where, for each $i = 1, \ldots, M$, B_i denotes a rectangular box in U^K with sides parallel to the sides of U^K, χ_{B_i} denotes the characteristic function of the box B_i, and the coefficients m_i are real.

THEOREM 2. *Suppose that $f \in \mathcal{F}(K)$. Then for every real number $W > 1$, there exists a positive constant $c_3 = c_3(f, W)$ such that for every M-simple function ϕ in U^K,*

$$\int_{U^K} |\phi(\mathbf{x}) - f(\mathbf{x})|^W \, d\mu > c_3(f, W) M^{-W} (\log M)^{\frac{1}{2}(K-1)W}.$$

As an immediate consequence, we have

COROLLARY 2. *Suppose that $f \in \mathcal{F}(K)$. Then there exists a positive constant $c_4 = c_4(f)$ such that for every M-simple function ϕ in U^K,*

$$\sup_{\mathbf{x} \in U^K} |\phi(\mathbf{x}) - f(\mathbf{x})| > c_4(f) M^{-1} (\log M)^{\frac{1}{2}(K-1)}.$$

We shall deduce Theorem 2 from Theorem 1 in § 2. In §§ 3–5, we shall prove Theorem 1. The method for the proof of Theorem 1 is based on Schmidt's adaptation [**14**] of Roth's auxiliary function method in [**9**]. However, Roth's argument in [**9**] makes use of the trivial fact that the function g, taken in [**9**] to be the constant function $g(\mathbf{y}) = 1$ in U^K, has constant sign and is bounded away from 0. To overcome the difficulty introduced by a function g which may take values with different signs, we make use of an observation by Schmidt in [**14**] which enables us to consider g only over a certain subset of U^K. We discuss this in § 4.

§ 2. Proof of Theorem 2

Let $f \in \mathcal{F}(K)$. Then

$$f(\mathbf{x}) = C + \int_{B(\mathbf{x})} g(\mathbf{y}) \, d\mu.$$

Clearly, we may assume, without loss of generality, that $C = 0$. Let

$$\phi(\mathbf{x}) = \sum_{i=1}^{M} m_i \chi_{B_i}(\mathbf{x}),$$

where, for each $i=1,\ldots,M$,

$$B_i=(u_1^{(i)},u_1^{(i)}+v_1^{(i)}]\times\cdots\times(u_K^{(i)},u_K^{(i)}+v_K^{(i)}]\subset U^K.$$

For each $i=1,\ldots,M$, let

$$\mathcal{P}_i=\{(u_1^{(i)}+\alpha_1v_1^{(i)},\ldots,u_K^{(i)}+\alpha_Kv_K^{(i)}):\ \alpha_1,\ldots,\alpha_K\in\{0,1\}\};$$

in other words, $\mathcal{P}_i$ is the set of vertices of the rectangular box B_i. For $\mathbf{y}\in U^K$ and $i=1,\ldots,M$, let

$$h_i(\mathbf{y})=\begin{cases}1 & (\mathbf{y}\in\mathcal{P}_i \text{ and } \alpha_1+\cdots+\alpha_K \text{ is even});\\ -1 & (\mathbf{y}\in\mathcal{P}_i \text{ and } \alpha_1+\cdots+\alpha_K \text{ is odd});\\ 0 & (\mathbf{y}\notin\mathcal{P}_i).\end{cases}$$

It is not difficult to see that

$$\chi_{B_i}(\mathbf{x})=\sum_{\mathbf{y}\in\mathcal{P}_i\cap B(\mathbf{x})}h_i(\mathbf{y})$$

in U^K.

Let

$$\mathcal{P}=\bigcup_{i=1}^{M}\mathcal{P}_i,$$

and let H be defined for all $\mathbf{y}\in U^K$ by

$$H(\mathbf{y})=\sum_{i=1}^{M}m_ih_i(\mathbf{y}).$$

Then clearly $\mathcal{P}$ is a distribution of $N\leq 2^KM$ points in U^K, and

$$\phi(\mathbf{x})=\sum_{\mathbf{y}\in\mathcal{P}\cap B(\mathbf{x})}H(\mathbf{y})=Z[\mathcal{P};H;B(\mathbf{x})]$$

in U^K.

Applying Theorem 1 with $h=MH$ and $N\leq 2^KM$ in (3), we get the desired result.

§ 3. An outline of the method of Roth and Schmidt

Following the method of Roth [**9**] and Schmidt [**14**], corresponding to every function of the type $D[\mathcal{P};h;g;B(\mathbf{x})]$, where $\mathcal{P}$ is a distribution of N points in U^K (N being sufficiently large), we construct an auxiliary function $F[\mathcal{P};h;g;\mathbf{x}]$ such that, writing $D(\mathbf{x})$ and $F(\mathbf{x})$ in place of $D[\mathcal{P};h;g;B(\mathbf{x})]$ and $F[\mathcal{P};h;g;\mathbf{x}]$ respectively,

(I) there exists a positive constant $c_5=c_5(g)$ such that

$$\int_{U^K}F(\mathbf{x})D(\mathbf{x})\,d\mu>c_5(g)(\log N)^{K-1};\tag{4}$$

and

(II) for every $m = 1, 2, \ldots$, there exists a positive constant $c_6 = c_6(g, m)$ such that

$$\int_{U^K} F^{2m}(\mathbf{x})\,\mathrm{d}\mu < c_6(g, m)(\log M)^{m(K-1)}. \tag{5}$$

Since $\left(\int_{U^K} |F(\mathbf{x})|^r\,\mathrm{d}\mu\right)^{1/r}$ is an increasing function of $r > 0$ for any fixed function F, we have, for every $r > 0$, that there exists a positive constant $c_7 = c_7(g, r)$ such that

$$\int_{U^K} |F(\mathbf{x})|^r\,\mathrm{d}\mu < c_7(g, r)(\log N)^{\frac{1}{2}(K-1)r}. \tag{6}$$

Theorem 1 follows from Hölder's inequality, (4) and (6), where r is chosen to satisfy $1/W + 1/r = 1$.

To prove Theorem 1, it therefore remains to show the existence of a function $F[\mathcal{P}; h; g; \mathbf{x}]$ that satisfies (I) and (II) above.

Some difficulty arises from the assumption that the function g can take different signs in any region. We therefore have to look for regions in U^K where g is "predominantly positive" or "predominantly negative". An observation in Schmidt [**14**] enables us to deal with the remaining regions in a trivial way. We discuss this in the next section.

§ 4. Preparation for the proof of Theorem 1

Let g be a Lebesgue-integrable function in U^K. Suppose that S is a measurable subset of U^K satisfying $\mu(S) > 0$ and $g(\mathbf{y}) \neq 0$ for every $\mathbf{y} \in S$. Then, replacing g by $-g$ if necessary, we may assume, without loss of generality, that there exist two positive constants $c_8 = c_8(g)$ and $c_9 = c_9(g)$ and a subset $S_1 \subset S$ such that

$$\mu(S_1) = c_8(g) \tag{7}$$

and

$$g(\mathbf{y}) \geq c_9(g) \text{ for every } \mathbf{y} \in S_1. \tag{8}$$

Consider the function

$$g^-(\mathbf{y}) = \max\{-g(\mathbf{y}), 0\}. \tag{9}$$

Then g^- is Lebesgue-integrable in U^K. Let

$$c_{10}(g) = 2^{-K-3}c_8(g)c_9(g). \tag{10}$$

Then there exists a positive constant $c_{11}=c_{11}(g)$ such that

$$c_{11}(g)\leq 2^{-K-4}c_8(g) \tag{11}$$

and such that for every measurable set $E\subset U^K$,

$$\int_E g^-(\mathbf{y})\,d\mu\leq c_{10}(g) \quad \text{if} \quad \mu(E)\leq c_{11}(g). \tag{12}$$

By an elementary box in U^K, we mean a set in U^K of the type

$$[m_1 2^{-t_1},(m_1+1)2^{-t_1})\times\cdots\times[m_K 2^{-t_K},(m_K+1)2^{-t_K}), \tag{13}$$

where $m_1,\ldots,m_K,t_1,\ldots,t_K$ are integers.

Consider the set S_1. Since S_1 is measurable, there exists a finite union T^* of elementary boxes in U^K such that

$$\mu(T^*\Delta S_1)\leq c_{11}(g),$$

where $T^*\Delta S_1$ denotes the symmetric difference of T^* and S_1. Hence if

$$E=T^*\backslash S_1, \tag{14}$$

then

$$\mu(E)\leq c_{11}(g). \tag{15}$$

Also, in view of (11), we see that

$$\mu(T^*)\geq \tfrac{1}{2}c_8(g). \tag{16}$$

Since T^* is a finite union of elementary boxes of the type (13), for every $j=1,\ldots,K$, there is one such elementary box with maximal t_j. Let T_j denote this maximal value of t_j, and let

$$T=T_1+\cdots+T_K. \tag{17}$$

We are now in a position to introduce the auxiliary function $F[\mathcal{P};h;g;\mathbf{x}]$.

Any $x\in[0,1)$ can be written in the form

$$x=\sum_{l=0}^{\infty}\beta_l(x)2^{-l-1},$$

where $\beta_l(x)=0$ or 1 such that the sequence $\beta_l(x)$ does not end with $1,1,\ldots$. For $r=0,1,\ldots,$ let

$$R_r(x)=(-1)^{\beta_r(x)}.$$

(These are called the Rademacher functions.)

DEFINITION. By an r-interval, we mean an interval of the form $[m2^{-r},(m+1)2^{-r})$, where the integer m satisfies $0\leq m<2^r$.

Suppose that $\mathbf{r}=(r_1,\ldots,r_K)$ is a K-tuple of non-negative integers. Let

$$|\mathbf{r}|=r_1+\cdots+r_K.$$

For any $\mathbf{x}\in[0,1)^K$, let

$$R_{\mathbf{r}}(\mathbf{x})=R_{r_1}(x_1)\cdots R_{r_K}(x_K).$$

DEFINITION. By an $\mathbf{r}$-box in U^K, we mean a set of the form $I_1\times\cdots\times I_K$, where, for $j=1,\ldots,K$, I_j is an r_j-interval.

In other words, these are special types of elementary boxes.

DEFINITION. By an $\mathbf{r}$-function, we mean a function $f(\mathbf{x})$ defined in $[0,1)^K$ such that in every $\mathbf{r}$-box, $f(\mathbf{x})=R_{\mathbf{r}}(\mathbf{x})$ or $f(\mathbf{x})=-R_{\mathbf{r}}(\mathbf{x})$.

LEMMA 1. *Suppose that $K\geq 2$, $m\geq 1$ and $n\geq 0$. Suppose further that for every $\mathbf{r}$ with $|\mathbf{r}|=n$, $f_{\mathbf{r}}$ is an $\mathbf{r}$-function. If*

$$F(\mathbf{x})=\sum_{|\mathbf{r}|=n}f_{\mathbf{r}}(\mathbf{x}),\tag{18}$$

then

$$\int_{U^K}F^{2m}(\mathbf{x})\,d\mu<(2m)^{m(2K-3)}(n+1)^{m(K-1)}.\tag{19}$$

Lemma 1 is precisely Lemma 4 of Schmidt [**14**]. We omit the proof here. Note that we have not chosen our auxiliary function $F[\mathcal{P};h;g;\mathbf{x}]$ yet, as (19) holds for any function of the form (18).

§ 5. Completion of the proof of Theorem 1

We shall only prove (4) and (5) for

$$N>\tfrac{1}{8}c_8(g)2^{2T}.\tag{20}$$

Let N satisfying (20) be given. Let n be a positive integer such that

$$\tfrac{1}{4}c_8(g)2^{n-1}<N\leq\tfrac{1}{4}c_8(g)2^n.\tag{21}$$

Then we have, in particular,

$$n\geq 2T.\tag{22}$$

By Lemma 1 and (21), we see that (5) holds. It therefore remains to prove (4).

LEMMA 2. *Suppose that* (21) *holds. Then for every $\mathbf{r}$ satisfying $|\mathbf{r}|=n$ and $r_j\geq T_j$ for every $j=1,\ldots,K$, there is an $\mathbf{r}$-function $f_{\mathbf{r}}$ satisfying*

$$\int_{U^K}f_{\mathbf{r}}(\mathbf{x})D(\mathbf{x})\,d\mu\geq 2^{-n-2K-4}c_8(g)c_9(g)N.\tag{23}$$

LEMMA 3. *Suppose that* (21) *holds. Then for every* $\mathbf{r}$ *satisfying* $|\mathbf{r}|=n$, *there is an* $\mathbf{r}$*-function* $f_{\mathbf{r}}$ *satisfying*

$$\int_{U^K} f_{\mathbf{r}}(\mathbf{x})D(\mathbf{x})\,d\mu \geqslant 0. \tag{24}$$

(4) follows easily from Lemmas 2 and 3. We construct $F(\mathbf{x})$ by (18), where, for every $\mathbf{r}$ satisfying the hypotheses of Lemma 2, $f_{\mathbf{r}}$ is chosen to satisfy (23). For the remaining $\mathbf{r}$, $f_{\mathbf{r}}$ is chosen to satisfy (24). Now, in view of (17), the number of K-tuples $\mathbf{r}$ satisfying the hypotheses of Lemma 2 is greater than $c_{12}(K)(n-T)^{K-1}$, where $c_{12}(K)$ is a positive constant. It follows, in view of (22), that

$$\int_{U^K} F(\mathbf{x})D(\mathbf{x})\,d\mu \geqslant c_8(g)c_9(g)c_{12}(K)2^{-n-2K-4}(n-T)^{K-1}N$$

$$\geqslant c_{13}(g)2^{-n}n^{K-1}N,$$

where $c_{13}(g)$ is a positive constant. This gives (4), in view of (21).

The proof of Lemma 3 is implicit in the proof of Lemma 2, so we only prove Lemma 2 here.

Proof of Lemma 2. We decompose the integral (23) into integrals over $\mathbf{r}$-boxes. We shall say that an $\mathbf{r}$-box is "good" if it is contained in T^* and does not contain any point of $\mathcal{P}$. Following Schmidt [**14**], for any $\mathbf{r}$-box which is not "good", we simply choose $f_{\mathbf{r}}(\mathbf{x})=R_{\mathbf{r}}(\mathbf{x})$ or $f_{\mathbf{r}}(\mathbf{x})=-R_{\mathbf{r}}(\mathbf{x})$ to make the integral $\int f_{\mathbf{r}}(\mathbf{x})D(\mathbf{x})\,d\mu$ over the $\mathbf{r}$-box non-negative. Let B be a "good" $\mathbf{r}$-box given by

$$B=[m_1 2^{-r_1},(m_1+1)2^{-r_1})\times\cdots\times[m_K 2^{-r_K},(m_K+1)2^{-r_K}),$$

and let B' be the box

$$B'=[m_1 2^{-r_1},(m_1+\tfrac{1}{2})2^{-r_1})\times\cdots\times[m_K 2^{-r_K},(m_K+\tfrac{1}{2})2^{-r_K}).$$

Then it is not difficult to see that $\int_B R_{\mathbf{r}}(\mathbf{x})D(\mathbf{x})\,d\mu$ is equal to

$$\int_{B'} \sum_{\alpha_1=0}^{1}\cdots\sum_{\alpha_K=0}^{1}(-1)^{\alpha_1+\cdots+\alpha_K}D((x_1+\alpha_1 2^{-r_1-1},\ldots,x_K+\alpha_K 2^{-r_K-1}))\,d\mu. \tag{25}$$

Note that, in view of (1), the sum

$$\left|\sum_{\alpha_1=0}^{1}\cdots\sum_{\alpha_K=0}^{1}(-1)^{\alpha_1+\cdots+\alpha_K}Z[\mathcal{P};h;B((x_1+\alpha_1 2^{-r_1-1},\ldots,x_K+\alpha_K 2^{-r_K-1}))]\right| = \left|\sum_{\mathbf{y}\in\mathcal{P}\cap B^*(\mathbf{x})} h(\mathbf{y})\right|, \quad (26)$$

where $B^*(\mathbf{x})=[x_1, x_1+2^{-r_1-1})\times\cdots\times[x_K, x_K+2^{-r_K-1})\subset B$ for every $\mathbf{x}\in B'$. Hence the sum (26) vanishes, and so, in view of (2), we have that (25) is equal to

$$-N\int_{B'}\left(\sum_{\alpha_1=0}^{1}\cdots\sum_{\alpha_K=0}^{1}(-1)^{\alpha_1+\cdots+\alpha_K}\int_0^{x_1+\alpha_1 2^{-r_1-1}}\cdots\int_0^{x_K+\alpha_K 2^{-r_K-1}} g(\mathbf{y})\,\mathrm{d}\mu\right)\mathrm{d}\mu$$

$$=(-1)^{K+1}N\int_{B'}\left(\int_{x_1}^{x_1+2^{-r_1-1}}\cdots\int_{x_K}^{x_K+2^{-r_K-1}} g(\mathbf{y})\,\mathrm{d}\mu\right)\mathrm{d}\mu$$

$$=(-1)^{K+1}N\int_B K_B(\mathbf{y})g(\mathbf{y})\,\mathrm{d}\mu,$$

where

$$K_B(\mathbf{y})=(2^{-r_1-1}-|y_1-(m_1+\tfrac{1}{2})2^{-r_1}|)\cdots(2^{-r_K-1}-|y_K-(m_K+\tfrac{1}{2})2^{-r_K}|).$$

For $\mathbf{x}\in B$, let $f_{\mathbf{r}}(\mathbf{x})=(-1)^{K+1}R_{\mathbf{r}}(\mathbf{x})$. Then for any "good" $\mathbf{r}$-box B,

$$\int_B f_{\mathbf{r}}(\mathbf{x})D(\mathbf{x})\,\mathrm{d}\mu = N\int_B K_B(\mathbf{y})g(\mathbf{y})\,\mathrm{d}\mu.$$

It follows that

$$\int_{U^K} f_{\mathbf{r}}(\mathbf{x})D(\mathbf{x})\,\mathrm{d}\mu \geqslant N\sum_{B\text{ "good"}}\int_B K_B(\mathbf{y})g(\mathbf{y})\,\mathrm{d}\mu.$$

In view of (16), for given $\mathbf{r}$ satisfying the hypotheses of Lemma 2, there are at least $(\tfrac{1}{2}c_8(g)2^n-N)$ "good" $\mathbf{r}$-boxes. It follows, by (8), (14), (9), (12), (21), (10), (15) and (11) that

$$\begin{aligned}\int_{U^K} f_{\mathbf{r}}(\mathbf{x})D(\mathbf{x})\,\mathrm{d}\mu &\geqslant Nc_9(g)\sum_{B\text{ "good"}}\int_B K_B(\mathbf{y})\,\mathrm{d}\mu - N2^{-n-K}c_9(g)\mu(E) - N2^{-n-K}\int_E g^-(\mathbf{y})\,\mathrm{d}\mu\\ &\geqslant Nc_9(g)2^{-2n-2K}(\tfrac{1}{2}c_8(g)2^n-N) - N2^{-n-K}c_9(g)\mu(E) - N2^{-n-K}c_{10}(g)\\ &\geqslant N2^{-n-K}(2^{-K-2}c_8(g)c_9(g)-c_9(g)\mu(E)-c_{10}(g))\\ &= N2^{-n-K}(2^{-K-3}c_8(g)c_9(g)-c_9(g)\mu(E))\\ &\geqslant N2^{-n-2K-4}c_8(g)c_9(g).\end{aligned}$$

This completes the proof of Lemma 2.

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