

DISCREPANCY FOR RANDOMIZED RIEMANN SUMS

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(Communicated by Michael T. Lacey)

ABSTRACT. Given a finite sequence $U_N = \{u_1, \dots, u_N\}$ of points contained in the d -dimensional unit torus, we consider the L^2 discrepancy between the integral of a given function and the Riemann sums with respect to translations of U_N . We show that with positive probability, the L^2 discrepancy of other sequences close to U_N in a certain sense preserves the order of decay of the discrepancy of U_N . We also study the role of the regularity of the given function.

1. INTRODUCTION

Let $N \in \mathbb{N}$ be a given large number, let $U_N = \{u_1, \dots, u_N\}$ be a distribution of N points in the unit cube $[-\frac{1}{2}, \frac{1}{2}]^d$, treated as the torus \mathbb{T}^d , and let f be a real function on \mathbb{T}^d . Suppose that for suitable choices of U_N and f , the Riemann sums

$$\frac{1}{N} \sum_{j=1}^N f(u_j - x)$$

are, after an L^2 average on the variable $x \in \mathbb{T}^d$, good approximations of the integral

$$\int_{\mathbb{T}^d} f(s) ds.$$

What corresponding statement can we make concerning those sequences *close* to the sequence U_N ? Do such sequences mostly share the same good behavior?

2. A RANDOMIZATION ARGUMENT

In order to start discussing these questions, we introduce the following randomization of U_N ; see [3, 6] and also [8, 9]. Let $d\mu$ denote a probability measure on \mathbb{T}^d . For every $j = 1, \dots, N$, let $d\mu_j$ denote the measure obtained after translating $d\mu$ by u_j . More precisely, for any integrable function g on \mathbb{T}^d , we have

$$\int_{\mathbb{T}^d} g(t) d\mu_j = \int_{\mathbb{T}^d} g(t - u_j) d\mu.$$

Received by the editors June 24, 2008.

2000 *Mathematics Subject Classification.* Primary 11K38; Secondary 41A55.

Key words and phrases. Irregularities of distribution, decay of Fourier coefficients.

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Let dt denote the Lebesgue measure on \mathbb{T}^d . For every sequence $V_N = \{v_1, \dots, v_N\}$ in \mathbb{T}^d and every function $f \in L^2(\mathbb{T}^d, dt)$, we introduce, for every $t \in \mathbb{T}^d$, the discrepancy

$$D(t, V_N) \stackrel{\text{def}}{=} \frac{1}{N} \sum_{j=1}^N f(v_j - t) - \int_{\mathbb{T}^d} f(s) ds.$$

Observe that $D(\cdot, V_N)$ is a periodic function with Fourier series

$$\sum_{0 \neq k \in \mathbb{Z}^d} \left(\frac{1}{N} \sum_{j=1}^N e^{-2\pi i k \cdot v_j} \right) \overline{\widehat{f}(k)} e^{2\pi i k \cdot t},$$

and the Parseval identity yields

$$D^2(V_N) \stackrel{\text{def}}{=} \|D(\cdot, V_N)\|_{L^2(\mathbb{T}^d, dt)}^2 = \sum_{0 \neq k \in \mathbb{Z}^d} \left| \frac{1}{N} \sum_{j=1}^N e^{2\pi i k \cdot v_j} \right|^2 |\widehat{f}(k)|^2.$$

We now average $D(V_N)$ in $L^2(\mathbb{T}^d, d\mu_j)$ for every $j = 1, \dots, N$ and consider

$$\mathfrak{D}_{d\mu}(U_N) \stackrel{\text{def}}{=} \left(\int_{\mathbb{T}^d} \dots \int_{\mathbb{T}^d} D^2(V_N) d\mu_1(v_1) \dots d\mu_N(v_N) \right)^{1/2}.$$

In this paper we study the relation between $\mathfrak{D}_{d\mu}(U_N)$ and $D(U_N)$. In the case $N = M^d$, where $M \in \mathbb{N}$, and

$$(1) \quad U_N = \frac{1}{M} \mathbb{Z}^d \cap \left[-\frac{1}{2}, \frac{1}{2} \right]^d,$$

the above quantities were studied in relation to the sharpness of a result of Beck [1] and of Montgomery [10] on irregularities of distribution; see Remark 3 below. In [6] two of the authors compared the quantities $D(U_N)$ and $\mathfrak{D}_{d\mu}(U_N)$ in the case (1) and when f is the characteristic function of a ball. Here we study the problem in our more general setting, and we are mainly interested in whether the inequality

$$(2) \quad \mathfrak{D}_{d\mu}(U_N) \leq c D(U_N)$$

holds.

Throughout this paper, the letters c, C, \dots will denote positive constants, possibly depending on f but independent of N , and which may change from one step to the next. On the other hand, different letters B, κ, \dots will denote constants which will not change throughout the paper.

3. AN EXPLICIT FORMULA

We first use a slight modification of an argument in [6] to obtain an explicit formula for $\mathfrak{D}_{d\mu}(U_N)$. We have

$$\begin{aligned}
(3) \quad \mathfrak{D}_{d\mu}^2(U_N) &= \int_{\mathbb{T}^d} \cdots \int_{\mathbb{T}^d} \sum_{0 \neq k \in \mathbb{Z}^d} \left| \frac{1}{N} \sum_{j=1}^N e^{2\pi i k \cdot v_j} \right|^2 |\widehat{f}(k)|^2 d\mu_1(v_1) \cdots d\mu_N(v_N) \\
&= \sum_{0 \neq k \in \mathbb{Z}^d} |\widehat{f}(k)|^2 \left(\frac{1}{N} + \frac{1}{N^2} \sum_{\substack{j, \ell=1 \\ j \neq \ell}}^N \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} e^{2\pi i k \cdot v_j} e^{-2\pi i k \cdot v_\ell} d\mu_j(v_j) d\mu_\ell(v_\ell) \right) \\
&= \sum_{0 \neq k \in \mathbb{Z}^d} |\widehat{f}(k)|^2 \\
&\quad \times \left(\frac{1}{N} + \frac{1}{N^2} \sum_{\substack{j, \ell=1 \\ j \neq \ell}}^N e^{2\pi i k \cdot (u_\ell - u_j)} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} e^{2\pi i k \cdot v_j} e^{-2\pi i k \cdot v_\ell} d\mu(v_j) d\mu(v_\ell) \right) \\
&= \sum_{0 \neq k \in \mathbb{Z}^d} |\widehat{f}(k)|^2 \left(\frac{1}{N} + |\widehat{\mu}(k)|^2 \left(\left| \frac{1}{N} \sum_{j=1}^N e^{2\pi i k \cdot u_j} \right|^2 - \frac{1}{N} \right) \right) \\
&= \frac{1}{N} \sum_{0 \neq k \in \mathbb{Z}^d} |\widehat{f}(k)|^2 (1 - |\widehat{\mu}(k)|^2) + \sum_{0 \neq k \in \mathbb{Z}^d} |\widehat{f}(k)|^2 |\widehat{\mu}(k)|^2 \left| \frac{1}{N} \sum_{j=1}^N e^{2\pi i k \cdot u_j} \right|^2 \\
&= \frac{1}{N} \left(\|f\|_{L^2(\mathbb{T}^d, dt)}^2 - \|f * d\mu\|_{L^2(\mathbb{T}^d, dt)}^2 \right) + \|D(\cdot, U_N) * d\mu\|_{L^2(\mathbb{T}^d, dt)}^2.
\end{aligned}$$

There are two natural extremal measures. The first one is $d\mu = \delta_0$, the Dirac measure centred at 0. In this case, we have

$$\mathfrak{D}_{\delta_0}(U_N) = D(U_N).$$

On the other hand, when $d\mu = dt$, we have

$$\mathfrak{D}_{dt}^2(U_N) = \frac{1}{N} \left(\|f\|_{L^2(\mathbb{T}^d, dt)}^2 - \left| \int_{\mathbb{T}^d} f(t) dt \right|^2 \right),$$

the classical Monte-Carlo error.

Note that if $ND^2(U_N) \geq c$, then $\mathfrak{D}_{dt}(U_N) \leq c_1 D(U_N)$, and (2) follows easily.

Another very peculiar case is when $D(U_N) = 0$. We observe that in general this does not imply $\mathfrak{D}_{d\mu}(U_N) = 0$, so that (2) does not hold. Indeed, let U_N be given by (1). Then

$$(4) \quad \frac{1}{N} \sum_{j=1}^N e^{2\pi i k \cdot u_j} = \begin{cases} 1 & \text{if } k \in M\mathbb{Z}^d, \\ 0 & \text{otherwise.} \end{cases}$$

Now choose $f(t) = \exp(2\pi i k_0 \cdot t)$ for some $k_0 \in \mathbb{Z}^d \setminus M\mathbb{Z}^d$. Then $D(U_N) = 0$. On the other hand, it follows from (3) that

$$\mathfrak{D}_{d\mu}^2(U_N) = \frac{1}{N}(1 - |\widehat{\mu}(k_0)|^2) \neq 0$$

whenever $|\widehat{\mu}(k_0)| \neq 1$, which is easily fulfilled, particularly by several measures with small support around the origin.

Hence, throughout the paper, we will be interested only in the case when

$$0 < D(U_N) < N^{-1/2}.$$

Let $0 < \varepsilon_N \leq 1$. For every probability measure $d\mu$ supported on the unit cube $[-\frac{1}{2}, \frac{1}{2}]^d$, let $d\mu^{(N)}$ denote the probability measure defined by

$$(5) \quad \int_{\mathbb{R}^d} g(\xi) d\mu^{(N)}(\xi) = \int_{\mathbb{R}^d} g(\varepsilon_N \xi) d\mu(\xi).$$

Then $d\mu^{(N)}$ is supported on the subcube $[-\frac{1}{2}\varepsilon_N, \frac{1}{2}\varepsilon_N]^d$ and can be regarded as a measure on \mathbb{T}^d .

4. MAIN RESULT

We first state our main result.

Theorem 1. *Let $f \in L^2(\mathbb{T}^d, dt)$ and let $U_N = \{u_1, \dots, u_N\}$ be a distribution of N points in the cube $[-\frac{1}{2}, \frac{1}{2}]^d$. Assume that $0 < D(U_N) < N^{-1/2}$. Let $d\mu$ be a non-Dirac probability measure on \mathbb{T}^d , let $d\mu^{(N)}$ be defined by (5) with $0 < \varepsilon_N \leq 1$, and let*

$$\eta_N = \begin{cases} \varepsilon_N^{2\alpha} & \text{if } \alpha < 1, \\ \varepsilon_N^2 \log(1 + \varepsilon_N^{-1}) & \text{if } \alpha = 1, \\ \varepsilon_N^2 & \text{if } \alpha > 1. \end{cases}$$

(i) *If for some $\alpha > 0$ and for every $\rho > 1$ we have*

$$(6) \quad \sum_{\rho \leq |k| < 2\rho} |\widehat{f}(k)|^2 \leq c \rho^{-2\alpha},$$

then

$$(7) \quad \mathfrak{D}_{d\mu^{(N)}}^2(U_N) \leq c \eta_N N^{-1} + D^2(U_N).$$

(ii) *If there exists an open cone¹ $\Omega \subseteq \mathbb{R}^d$ such that for every subcone $\Gamma \subseteq \Omega$,*

$$(8) \quad \liminf_{\rho \rightarrow +\infty} \rho^{2\alpha} \sum_{\substack{k \in \Gamma \\ \rho \leq |k| < 2\rho}} |\widehat{f}(k)|^2 > 0,$$

then there exist positive constants $\Delta \leq 1$ and c such that if $\varepsilon_N \leq \Delta$, then

$$\mathfrak{D}_{d\mu^{(N)}}^2(U_N) \geq c \eta_N N^{-1}.$$

The following corollary shows that, in some sense, good sequences are never alone. Indeed we give conditions on ε_N that will ensure that $\mathfrak{D}_{d\mu^{(N)}}(U_N)$ and $D(U_N)$ are comparable.

Corollary 2. *Let f , U_N and $d\mu$ be as given in Theorem 1.*

¹In this paper every cone starts from the origin.

(i) Let f be as given in part (i) of Theorem 1 and let

$$(9) \quad \varepsilon_N \leq \begin{cases} (N^{1/2}D(U_N))^{1/\alpha} & \text{if } \alpha < 1, \\ \beta_N & \text{if } \alpha = 1, \\ N^{1/2}D(U_N) & \text{if } \alpha > 1, \end{cases}$$

where β_N satisfies $\beta_N^2 \log(1 + \beta_N^{-1}) = ND^2(U_N)$. Then

$$\mathfrak{D}_{d\mu^{(N)}}^2(U_N) \leq cD^2(U_N).$$

(ii) Let f and Δ be as given in part (ii) of Theorem 1 and let $\kappa > 0$. Then there exists $c > 0$ such that whenever

$$(10) \quad \Delta \geq \varepsilon_N \geq \begin{cases} \kappa(N^{1/2}D(U_N))^{1/\alpha} & \text{if } \alpha < 1, \\ \kappa\beta_N & \text{if } \alpha = 1, \\ \kappa N^{1/2}D(U_N) & \text{if } \alpha > 1, \end{cases}$$

we have

$$\mathfrak{D}_{d\mu^{(N)}}^2(U_N) \geq cD^2(U_N).$$

Remark 3. Consider the particular case when $f = \chi_A$, the characteristic function of a convex body $A \subseteq [-\frac{1}{2}, \frac{1}{2}]^d$. Then (6) holds with $\alpha = \frac{1}{2}$. Let $\varepsilon_N = \Delta ND^2(U_N)$. Then

$$\mathfrak{D}_{d\mu^{(N)}}^2(U_N) \leq cD^2(U_N).$$

If furthermore the boundary of A is smooth and has positive Gaussian curvature, then (8) holds with $\alpha = \frac{1}{2}$; see, for instance, [7]. We then have

$$\mathfrak{D}_{d\mu^{(N)}}^2(U_N) \geq cD^2(U_N).$$

We recall that if A is rotated and contracted, then a result of Beck [1] and of Montgomery [10] says that

$$\int_{SO(d)} \int_0^1 \int_{\mathbb{T}^d} \left| \frac{1}{N} \sum_{j=1}^N \chi_{\sigma(rA)}(u_j - t) - r^d |A| \right|^2 dt dr d\sigma \geq cN^{-1-1/d}$$

for every choice of the point set distribution U_N ; see also [2, 4, 5]. We also recall that this is not true if the contraction is omitted; see [12, Theorem 3.1].

5. DECAY OF THE FOURIER COEFFICIENTS

The assumption (6) concerns the decay of the Fourier coefficients of f . This behavior can be naturally related to the smoothness of the function f as follows. Let $f \in L^2(\mathbb{T}^d)$, define $\Delta_h f(x) = f(x+h) - f(x)$ and, for every integer $\ell \geq 1$, write $\Delta_h^\ell f = \Delta_h \Delta_h^{\ell-1} f$. Let $\alpha > 0$. We say that f belongs to the Nikol'skiĭ space $H_2^\alpha(\mathbb{T}^d)$ if there exists $c > 0$ such that

$$\left(\int_{\mathbb{T}^d} |\Delta_h^\ell f(x)|^2 dx \right)^{1/2} \leq c|h|^\alpha$$

for some $\ell \geq 1$; see [11, Section 4.3.3].

Proposition 4. *Let $f \in H_2^\alpha(\mathbb{T}^d)$. Then (6) holds.*

Proof. Since $\widehat{\Delta_h f}(k) = (e^{2\pi i k \cdot h} - 1)\widehat{f}(k)$, we have $\widehat{\Delta_h^\ell f}(k) = (e^{2\pi i k \cdot h} - 1)^\ell \widehat{f}(k)$. Let $h = (1/10\rho, 0, \dots, 0)$ and $\Gamma = \{k \in \mathbb{Z}^d : k_1^2 \geq k_2^2 + \dots + k_d^2\}$. Observe that when $k \in \Gamma$ and $\rho \leq |k| \leq 2\rho$, we have $|e^{2\pi i k \cdot h} - 1| \geq c$. Therefore

$$\begin{aligned} \sum_{\substack{k \in \Gamma \\ \rho \leq |k| < 2\rho}} |\widehat{f}(k)|^2 &\leq c \sum_{\substack{k \in \Gamma \\ \rho \leq |k| < 2\rho}} |(e^{2\pi i k \cdot h} - 1)^\ell|^2 |\widehat{f}(k)|^2 \leq c \sum_{k \in \mathbb{Z}^d} |\widehat{\Delta_h^\ell f}(k)|^2 \\ &= c \int_{\mathbb{T}^d} |\Delta_h^\ell f(x)|^2 dx \leq c |h|^{2\alpha} = c \rho^{-2\alpha}. \end{aligned}$$

Note here that h is tailored on Γ . Since we can cover \mathbb{Z}^d with a finite number of cones, the proposition follows from the above argument applied to different choices of h . \square

The proof of Theorem 1 requires a technical lemma.

Lemma 5. *Let $d\nu$ be a probability measure supported on $[-\frac{1}{2}, \frac{1}{2}]^d$. Then either*

- (i) *$d\nu$ is the Dirac measure δ_{t_0} at a point $t_0 \in [-\frac{1}{2}, \frac{1}{2}]^d$, or*
- (ii) *$1 - |\widehat{\nu}(\xi)|^2 = O(|\xi|^2)$ as $\xi \rightarrow 0$, and any open cone in \mathbb{R}^d contains an open subcone Γ such that $1 - |\widehat{\nu}(\xi)|^2 \geq c|\xi|^2$ for small $\xi \in \Gamma$.*

Proof. Since $d\nu$ is compactly supported, its Fourier transform $\widehat{\nu}$ is smooth and has Taylor expansion

$$\widehat{\nu}(\xi) = 1 + \nabla \widehat{\nu}(0) \cdot \xi + \frac{1}{2} H_{\widehat{\nu}}(0) \xi \cdot \xi + o(|\xi|^2),$$

and so

$$1 - |\widehat{\nu}(\xi)|^2 = 1 - \widehat{\nu}(\xi)\widehat{\nu}(-\xi) = (\nabla \widehat{\nu}(0) \cdot \xi)^2 - H_{\widehat{\nu}}(0) \xi \cdot \xi + o(|\xi|^2) = O(|\xi|^2).$$

Let $F(\xi) = (\nabla \widehat{\nu}(0) \cdot \xi)^2 - H_{\widehat{\nu}}(0) \xi \cdot \xi$, and assume that F does not vanish identically. Let $\Sigma_{d-1} = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$. Since F is a polynomial, it cannot vanish on an open set, and therefore $\{\xi \in \Sigma_{d-1} : F(\xi) = 0\}$ has empty interior in Σ_{d-1} . Since F is homogeneous and continuous, it follows that for every open cone in \mathbb{R}^d , we can find an open subcone Γ such that $|F(\xi)| \geq c|\xi|^2$ for $\xi \in \Gamma$. Therefore $1 - |\widehat{\nu}(\xi)|^2 \geq c|\xi|^2$ for small $\xi \in \Gamma$.

Assume now that $F \equiv 0$. Observe that

$$\frac{\partial \widehat{\nu}}{\partial \xi_j}(0) = -2\pi i \int_{\mathbb{T}^d} x_j d\nu(x)$$

and

$$\frac{\partial^2 \widehat{\nu}}{\partial \xi_j \partial \xi_\ell}(0) = -4\pi^2 \int_{\mathbb{T}^d} x_j x_\ell d\nu(x).$$

Then

$$\nabla \widehat{\nu}(0) \cdot \xi = -2\pi i \int_{\mathbb{T}^d} (x \cdot \xi) d\nu(x)$$

and

$$H_{\widehat{\nu}}(0) \xi \cdot \xi = -4\pi^2 \sum_{i,j} \int_{\mathbb{T}^d} \xi_j \xi_\ell x_j x_\ell d\nu(x) = -4\pi^2 \int_{\mathbb{T}^d} (\xi \cdot x)^2 d\nu(x).$$

Hence

$$\begin{aligned} 0 &= (\nabla \widehat{\nu}(0) \cdot \xi)^2 - H_{\widehat{\nu}}(0) \xi \cdot \xi = -4\pi^2 \left(\int_{\mathbb{T}^d} (x \cdot \xi) \, d\nu(x) \right)^2 + 4\pi^2 \int_{\mathbb{T}^d} (\xi \cdot x)^2 \, d\nu(x) \\ &= 4\pi^2 \int_{\mathbb{T}^d} \left(x \cdot \xi - \int_{\mathbb{T}^d} (t \cdot \xi) \, d\nu(t) \right)^2 \, d\nu(x). \end{aligned}$$

Let

$$t_0 = \int_{\mathbb{T}^d} t \, d\nu(t).$$

Since $d\nu(x)$ is positive, it follows that for every fixed ξ , we have

$$\nu(\{x : x \cdot \xi - \xi \cdot t_0 \neq 0\}) = 0.$$

Since ξ is arbitrary, we conclude that $\nu(\{x : x - t_0 \neq 0\}) = 0$, so that $d\nu$ is supported at t_0 . Since $d\nu$ is a probability measure, we have $d\nu = \delta_{t_0}$. \square

6. PROOF OF THEOREM 1

By Lemma 5, we have

$$1 - |\widehat{\mu}^{(N)}(k)|^2 = 1 - |\widehat{\mu}(\varepsilon_N k)|^2 = O(\varepsilon_N^2 |k|^2).$$

As $d\mu$ is a probability measure, we have

$$0 \leq 1 - |\widehat{\mu}^{(N)}(k)|^2 \leq \min\{1, c\varepsilon_N^2 |k|^2\}.$$

By (6), we have

$$\begin{aligned} (11) \quad \sum_{k \in \mathbb{Z}^d} |\widehat{f}(k)|^2 (1 - |\widehat{\mu}^{(N)}(k)|^2) &\leq \sum_{k \in \mathbb{Z}^d} |\widehat{f}(k)|^2 \min\{1, c\varepsilon_N^2 |k|^2\} \\ &\leq \sum_{j=0}^{+\infty} \min\{1, c\varepsilon_N^2 2^{2j}\} \sum_{2^j \leq |k| < 2^{j+1}} |\widehat{f}(k)|^2 \leq c \sum_{j=0}^{+\infty} \min\{1, \varepsilon_N^2 2^{2j}\} 2^{-2j\alpha} \\ &\leq c\varepsilon_N^2 \sum_{2^j < \varepsilon_N^{-1}} 2^{(2-2\alpha)j} + c \sum_{2^j > \varepsilon_N^{-1}} 2^{-2j\alpha}. \end{aligned}$$

There are three cases. If $\alpha < 1$, we have

$$\sum_{0 \neq k \in \mathbb{Z}^d} |\widehat{f}(k)|^2 (1 - |\widehat{\mu}^{(N)}(k)|^2) \leq c\varepsilon_N^{2\alpha}.$$

If $\alpha = 1$, we have

$$\sum_{0 \neq k \in \mathbb{Z}^d} |\widehat{f}(k)|^2 (1 - |\widehat{\mu}^{(N)}(k)|^2) \leq c\varepsilon_N^2 \log(1 + \varepsilon_N^{-1}).$$

If $\alpha > 1$, we have

$$\sum_{0 \neq k \in \mathbb{Z}^d} |\widehat{f}(k)|^2 (1 - |\widehat{\mu}^{(N)}(k)|^2) \leq c\varepsilon_N^2.$$

Since $d\mu$ is a probability measure, we have

$$(12) \quad \|D(\cdot, U_N) * d\mu\|_{L^2(\mathbb{T}^d, dt)} \leq D(U_N).$$

In view of (11) and (12), the inequality (7) follows from (3).

Let us now prove (ii). By Lemma 5 there exists a subcone $\Gamma \subset \Omega$ such that $1 - |\widehat{\mu}(\xi)|^2 \geq m_1|\xi|^2$ for $|\xi| \leq m_2, \xi \in \Gamma$. By (8) there exist m_3 and m_4 such that for $\rho \geq m_3$ we have

$$\sum_{\substack{k \in \Gamma \\ \rho \leq |k| < 2\rho}} |\widehat{f}(k)|^2 \geq m_4 \rho^{-2\alpha}.$$

Thus, for $\varepsilon_N < \min\{m_2/4m_3, 1\}$, we have

$$\begin{aligned} \mathfrak{D}_{d\mu^{(N)}}^2(U_N) &\geq \frac{1}{N} \left(\|f\|_{L^2(\mathbb{T}^d, dt)}^2 - \|f * d\mu^{(N)}\|_{L^2(\mathbb{T}^d, dt)}^2 \right) \\ &= \frac{1}{N} \sum_{0 \neq k \in \mathbb{Z}^d} |\widehat{f}(k)|^2 (1 - |\widehat{\mu}(\varepsilon_N k)|^2) \\ &\geq \frac{1}{N} \sum_{m_3 \leq 2^j \leq \frac{1}{2} m_2 \varepsilon_N^{-1}} \sum_{\substack{k \in \Gamma \\ 2^j \leq |k| < 2^{j+1}}} |\widehat{f}(k)|^2 (1 - |\widehat{\mu}(\varepsilon_N k)|^2) \\ &\geq \frac{\varepsilon_N^2}{N} m_1 m_4 \sum_{m_3 \leq 2^j \leq \frac{1}{2} m_2 \varepsilon_N^{-1}} 2^{-2j\alpha} 2^{2j} \geq c \eta_N N^{-1}. \end{aligned}$$

This completes the proof of Theorem 1.

Remark 6. The estimates from below for $\mathfrak{D}_{d\mu^{(N)}}^2(U_N)$ contained in Theorem 1 and Corollary 2 depend on suitable estimates for the first term

$$\frac{1}{N} \left(\|f\|_{L^2(\mathbb{T}^d, dt)}^2 - \|f * d\mu^{(N)}\|_{L^2(\mathbb{T}^d, dt)}^2 \right)$$

in (3). We observe that in our setting the second term may vanish even in rather natural examples. Indeed, let

$$f(x) = \sum_{k \neq 0} \frac{1}{|k|^\gamma} e^{2\pi i k x}$$

for some $\gamma > d/2 + 1$. One can easily check that (8) holds with $\alpha = \gamma - d/2$. Let U_N be as in (1) and μ be the (normalized) Lebesgue measure restricted to $[-\frac{1}{2}, \frac{1}{2}]^d$, so that, taking $\varepsilon_N = 1/M$, we have

$$\widehat{\mu^{(N)}}(k) = N \prod_{j=1}^d \frac{\sin(\pi k_j / M)}{\pi k_j}.$$

By (4) we have

$$D^2(U_N) = \sum_{k \neq 0} |\widehat{f}(Mk)|^2 = \frac{1}{M^{2\gamma}} \sum_{k \neq 0} \frac{1}{|k|^{2\gamma}} = \frac{c_\gamma}{M^{2\gamma}}$$

and

$$\|D(\cdot, U_N) * d\mu^{(N)}\|_{L^2(\mathbb{T}^d, dt)} = \sum_{k \neq 0} |\widehat{f}(Mk)|^2 |\widehat{\mu^{(N)}}(Mk)|^2 = 0.$$

On the other hand observe that, for large N ,

$$\varepsilon_N = \frac{1}{M} \geq c_\gamma N^{\frac{1}{2}} D(U_N) = c_\gamma M^{d/2-\gamma},$$

and therefore we can apply part (ii) of Corollary 2 and obtain the inequality $\mathfrak{D}_{d\mu^{(N)}}^2(U_N) \geq c D^2(U_N)$.

7. CONCLUSION

Let $d\mu^{\otimes}$ be defined on $(\mathbb{T}^d)^N$ by

$$\int_{(\mathbb{T}^d)^N} \varphi d\mu^{\otimes} = \int_{\mathbb{T}^d} \dots \int_{\mathbb{T}^d} \varphi(v_1 - u_1, \dots, v_N - u_N) d\mu^{(N)}(v_1) \dots d\mu^{(N)}(v_N).$$

We can now state and prove the result introduced in the abstract.

Corollary 7. *Let f , U_N and $d\mu$ be as given in Corollary 2.*

- (i) *Let f and ε_N be as given in part (i) of Corollary 2. Then for every λ satisfying $0 < \lambda < 1$, there exists a constant $c_\lambda > 0$, independent of U_N and such that $d\mu^{\otimes}(\{V_N : D(V_N) \leq c_\lambda D(U_N)\}) \geq \lambda$.*
- (ii) *Let f , Δ and ε_N be as given in part (ii) of Corollary 2. Then for a suitable constant $c > 0$, we have $d\mu^{\otimes}(\{V_N : D(V_N) \geq cD(U_N)\}) > 0$.*

Proof. If (9) holds, then Corollary 2 gives

$$\int_{\mathbb{T}^d} \dots \int_{\mathbb{T}^d} D^2(V_N) d\mu^{\otimes}(V_N) \leq cD^2(U_N).$$

By the Chebyshev inequality, we have

$$d\mu^{\otimes}(\{V_N : D(V_N) > c_\lambda D(U_N)\}) \leq \frac{c}{c_\lambda^2},$$

and so

$$d\mu^{\otimes}(\{V_N : D(V_N) \leq c_\lambda D(U_N)\}) \geq 1 - \frac{c}{c_\lambda^2}.$$

A suitable choice of c_λ completes the proof of part (i). If (10) and (8) hold, then Corollary 2 gives

$$\int_{\mathbb{T}^d} \dots \int_{\mathbb{T}^d} D^2(V_N) d\mu^{\otimes}(V_N) \geq cD^2(U_N),$$

which easily implies $d\mu^{\otimes}(\{V_N : D(V_N) \geq cD(U_N)\}) > 0$. □

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