A MOD N VERSION OF THE KRONECKER–WEYL EQUIDISTRIBUTION THEOREM

J. BECK AND W.W.L. CHEN

Abstract. We establish various analogs of the Kronecker–Weyl equidistribution theorem that can be considered higher-dimensional versions of results established in our earlier investigation in [1] of the discrete 2-circle problem studied in 1969 by Veech [4]. Whereas the Veech problem can be viewed as one of geodesic flow on a 2-dimensional flat surface, here we study geodesic flow in higher-dimensional flat manifolds. This is more challenging, as the overwhelming majority of the available proof techniques for non-integrable flat systems are based on arguments in dimension 2. For higher dimensions, we need a new approach.

1. Introduction

Our starting point is the multi-dimensional version of the equidistribution of the irrational rotation sequence. The Kronecker–Weyl equidistribution theorem is a quintessential result for flat dynamical systems.

Let the dimension $d \geqslant 2$ be fixed. We say that a vector

$$\mathbf{v} = (1, \alpha_1, \ldots, \alpha_{d-1}) \in \mathbb{R}^d \tag{1.1}$$

is a Kronecker vector if the d coordinates $1, \alpha_1, \ldots, \alpha_{d-1}$ are linearly independent over $\mathbb{Q}$, so that the only solution of the equation

$$a_0 + a_1\alpha_1 + \ldots + a_{d-1}\alpha_{d-1} = 0, \quad a_0, a_1, \ldots, a_{d-1} \in \mathbb{Q},$$

is the trivial solution $a_0 = a_1 = \ldots = a_{d-1} = 0$. We also write

$$\mathbf{v}_0 = (\alpha_1, \ldots, \alpha_{d-1}) \in \mathbb{R}^{d-1}, \tag{1.2}$$

so that $\mathbf{v} = (1, \mathbf{v}_0)$.

Theorem (Kronecker–Weyl). *Let the integer $d \geqslant 2$ be fixed.*

(i) *Let $\mathcal{L}(\mathbf{v}, t)$, $t \geqslant 0$, be a half-infinite 1-direction geodesic on the d-dimensional unit torus $[0,1)^d$, with arc-length parametrization, and with direction vector $\mathbf{v}$ given by* (1.1) *and arbitrary starting point $\mathcal{L}(\mathbf{v}, 0)$. Then this geodesic is dense in $[0,1)^d$ if and only if $\mathbf{v}$ is a Kronecker vector.*

(ii) *Let $\mathbf{s} + n\mathbf{v}_0$, $n = 0, 1, 2, 3, \ldots$, be an infinite arithmetic progression in the $(d-1)$-dimensional unit torus $[0,1)^{d-1}$, with shift vector $\mathbf{v}_0$ given by* (1.1) *and* (1.2) *and arbitrary starting point $\mathbf{s}$. Then this arithmetic progression is dense in $[0,1)^{d-1}$ if and only if $\mathbf{v}$ is a Kronecker vector.*

(iii) *The geodesic in part* (i) *is equidistributed in $[0,1)^d$ if and only if $\mathbf{v}$ is a Kronecker vector.*

(iv) *The arithmetic progression in part* (ii) *is equidistributed in $[0,1)^{d-1}$ if and only if $\mathbf{v}$ is a Kronecker vector.*

(v) *Let $\mathcal{S}(\mathbf{v}, t)$, $t \geqslant 0$, be a half-infinite billiard orbit in the d-dimensional unit cube $[0,1]^d$, with arc-length parametrization, and initial direction vector $\mathbf{v}$ given by* (1.1) *and arbitrary starting point $\mathcal{S}(\mathbf{v}, 0)$. Then this orbit is equidistributed in $[0,1]^d$ if and only if $\mathbf{v}$ is a Kronecker vector.*

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Parts (i) and (ii) are well known results of Kronecker, while the equally famous extensions (iii) and (iv) are due to Weyl. For more details, the reader is referred to the treatise [2].

Part (v) follows from combining part (iii) with the geometric trick of *unfolding* due to König and Szücs [3], as illustrated in Figure 1.1 in the simplest 2-dimensional case.

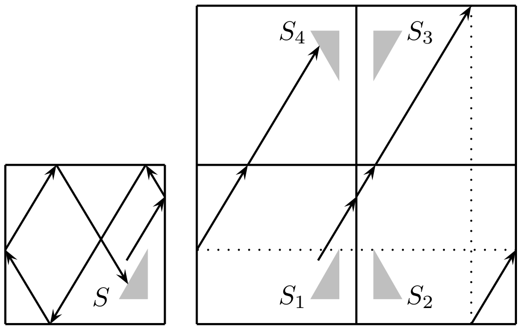

Figure 1.1: unfolding of the square billiard

In Figure 1.1, the 2×2 torus in the picture on the right is a 4-fold covering of the unit square in the picture on the left, with the covering map

$$(x, y) \to (\min\{2-x, x\}, \min\{2-y, y\}), \quad (x, y) \in [0, 2)^2,$$

where $0 \leqslant \{z\} < 1$ denotes the fractional part of a real number z. Clearly this covering map projects a 1-direction geodesic on the 2×2 torus to a billiard orbit in the unit square $[0, 1]^2$.

Since we can tile the whole plane with 2×2 squares, the König–Szücs construction has the remarkable property that it unfolds the zig-zag billiard orbit in the square $[0, 1]^2$ to a straight line on the plane.

Note also that unfolding works in any dimension $d \geqslant 3$, where we replace the unit square with the d-dimensional unit cube $[0, 1]^d$. We have a 2^d-fold covering of the unit cube $[0, 1]^d$.

For completeness, we recall the definition of equidistribution.

A half-infinite 1-direction geodesic is equidistributed, or uniformly distributed, in the unit torus $[0, 1)^d$ if and only if for any subcube $Q \subset [0, 1)^d$, the asymptotic relative time the geodesic spends in Q is equal to the d-dimensional volume of Q.

We have analogous definitions of equidistribution for billiard orbits and for discrete infinite sequences.

We extend the idea of Veech [4] to higher dimension. First we 2-color the unit torus $[0, 1)^2$ red and green in such a way that each of the red and green parts is the union of finitely many polygons. Figure 1.2 shows two examples, where the shaded part represents red and the white part represents green. In particular, we assume that the green part has positive area.

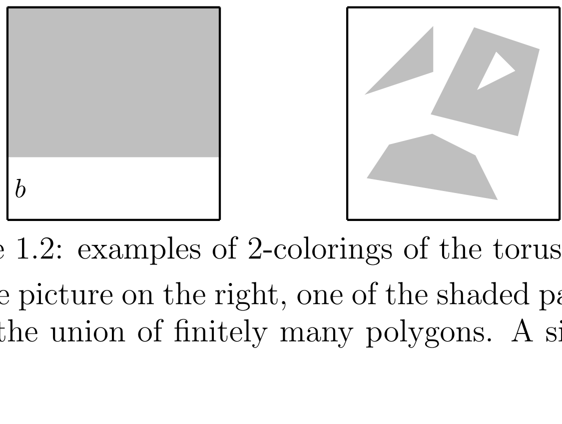

Figure 1.2: examples of 2-colorings of the torus $[0, 1)^2$

Observe that in the picture on the right, one of the shaded parts does not look like a polygon, but it is the union of finitely many polygons. A similar remark applies to the white part.

Next, we consider a 2-torus system as shown in Figure 1.3, where each square represents the unit torus $[0,1)^2$, with identical 2-coloring.

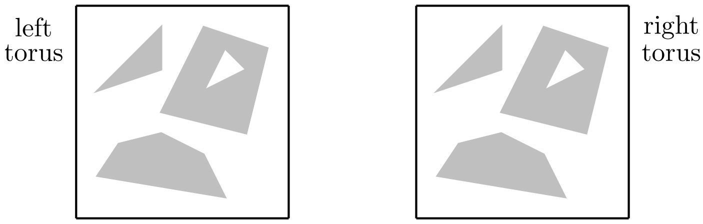

Figure 1.3: a 2-torus system with identical coloring

Let $\mathbf{v} = (1, \alpha_1, \alpha_2)$ be a Kronecker vector, and let $\mathbf{v}_0 = (\alpha_1, \alpha_2)$.

Let $\mathbf{s} \in [0,1)^2$ be an arbitrary starting point, and consider the $\mathbf{v}_0$-shift sequence

$$\mathbf{s}_n = \mathbf{s} + n\mathbf{v}_0, \quad n = 0, 1, 2, 3, \ldots,$$

in the unit torus $[0,1)^2$; in other words, *modulo one*. Assume that the point $\mathbf{s}_0$ is on the left torus. If $\mathbf{s}_1$ is in the red (shaded) part, then we keep it on the left torus. If $\mathbf{s}_1$ is in the green (white) part, then we move it to the corresponding point on the right torus. In general, $\mathbf{s}_n$ is on a particular torus. If $\mathbf{s}_{n+1}$ is in the red (shaded) part, then we keep it on the same torus. If $\mathbf{s}_{n+1}$ is in the green (white) part, then we move it to the corresponding point on the other torus. Thus the sequence $\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \ldots$ moves from one torus to the other whenever it hits the red part. The problem is then to describe the distribution of this sequence in the union of the two tori, clearly a *parity* problem motivated by the Kronecker–Weyl equidistribution theorem.

We can visualize this discrete 2-torus system on the plane as a simple continuous system in 3-space. Figure 1.4 illustrates this observation in the case of the simpler 2-coloring in the picture on the left in Figure 1.2.

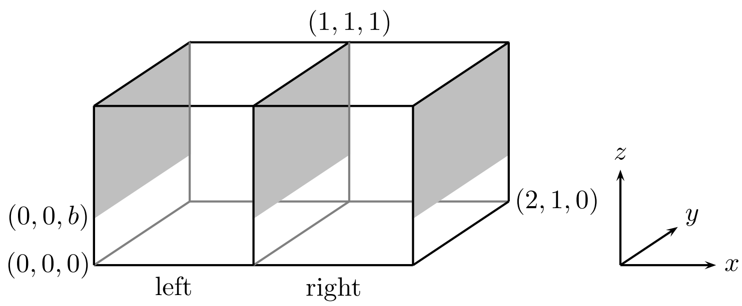

Figure 1.4: 2-cube-b manifold with repeated barriers

Here there are three yz-parallel square faces of the 2-cube solid, each of which is in part a barrier, colored red (shaded), and a gate, colored green (white). The latter is non-empty, and permits travel between the two cubes. To complete the requirements for this to be a 3-manifold, we have to guarantee that it is boundary-free. We use boundary identification which is a modification of the boundary identification for the torus $[0,1)^3$. The two xy-parallel square faces with $z = 0$ are identified with the two xy-parallel square faces with $z = 1$ by trivial perpendicular translation. The two xz-parallel square faces with $y = 0$ are identified with the two xz-parallel square faces with $y = 1$ by trivial perpendicular translation. The right side of the red (shaded) rectangle on the yz-parallel square face with $x = 0$ is identified with the left side of the red (shaded) rectangle on the yz-parallel square face with $x = 1$, while the right side of the red (shaded) rectangle on the yz-parallel square face with $x = 1$ is identified with the left side of the red (shaded) rectangle on the yz-parallel square face with $x = 2$. Finally, the green (white) rectangle on the yz-parallel square face with $x = 0$ is identified with the green (white) rectangle on the yz-parallel square face with $x = 2$.

For convenience, we refer to this as the 2-cube-b manifold.

We thus have a flat 3-manifold, with euclidean metric almost everywhere, and with boundary identification via perpendicular translation. Thus geodesic flow on this manifold is 1-direction linear flow. It moves rather like 1-direction geodesic flow on the torus $[0,1)^3$, and the novelty comes from the effect of the barriers.

There is clearly an equivalence between the discrete 2-dimensional 2-torus system and this new continuous 3-dimensional 2-cube system. An infinite $\mathbf{v}_0$-shift sequence is equidistributed on the 2-torus with the 2-coloring given in the picture on the left of Figure 1.2 if and only if the corresponding half-infinite 1-direction geodesic with direction vector $\mathbf{v}$ is equidistributed in the 2-cube-b manifold.

Assume now that $\mathbf{v}=(1,\alpha_1,\alpha_2)\in\mathbb{R}^3$ is a Kronecker vector. Is it true that every half-infinite 1-direction geodesic with direction vector $\mathbf{v}$ is equidistributed in any 2-cube-b manifold with $0<b<1$?

It turns out that for every Kronecker vector $\mathbf{v}$, there are infinitely many values of the parameter b for which equidistribution fails. To explain this, we need to look at the corresponding problem in lower dimension. The projection of the 2-cube-b manifold to the xz-plane gives rise to the 2-square-b surface which arises from the work of Veech [4]. Some of the anti-equidistribution results on such surfaces obtained recently by the authors and Yang in [1] can be converted to anti-equidistribution results on 2-cube-b manifolds.

For instance, let $\alpha_2\in(0,1/2)$ be irrational, and let $b=2\alpha_2$. Then for every α_1 for which $\mathbf{v}=(1,\alpha_1,\alpha_2)\in\mathbb{R}^3$ is a Kronecker vector, every half-infinite 1-direction geodesic with direction vector $\mathbf{v}$ violates equidistribution in the 2-cube-b manifold. For more details, see [1, Theorem 2.1].

The papers [1] and [4] contain several equidistribution results on the 2-square-b surface. These, however, do not give corresponding results on the 2-cube-b manifold. Nevertheless, using a different approach, we can establish equidistribution for most half-infinite geodesics in the 2-cube-b manifold. Furthermore, we can generalize the result to any 2-coloring of the unit torus $[0,1)^2$ where each of the red and green parts is the union of finitely many polygons, and where the green part has positive area. The richness of the possibilities to fix such 2-colorings is particularly interesting.

Indeed, we can consider an n-torus system, with n copies of the unit torus $[0,1)^2$, where $n\geqslant 2$ is an integer. This then leads to a flat 3-manifold, with euclidean metric almost everywhere, and with boundary identification via perpendicular translation. For instance, if we take $n=4$ and use the 2-coloring of the torus $[0,1)^2$ as shown in the picture on the right in Figure 1.2, then we have the 4-cube 3-manifold with repeated barriers as shown in Figure 1.5.

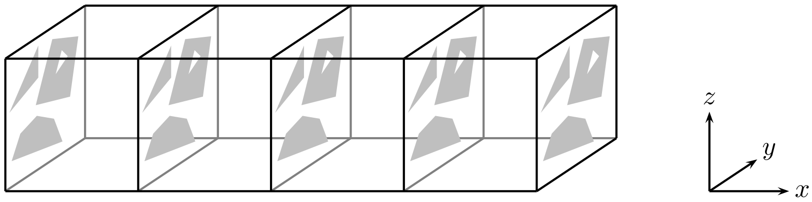

Figure 1.5: a 4-cube 3-manifold with repeated barriers

Theorem 1.1. *Let $n\geqslant 2$ be an integer, and let $\mathcal{P}$ be any n-cube 3-manifold with repeated barriers, where the $n+1$ yz-parallel square faces have the same 2-coloring such that each of the red and green parts is the union of finitely many polygons, and where the green part has positive area. Then for almost every starting point and for almost every direction $\mathbf{v}=(1,\alpha_1,\alpha_2)\in\mathbb{R}^3$ with $\mathbf{v}_0=(\alpha_1,\alpha_2)\in[-1,1]^2$, the corresponding half-infinite 1-direction geodesic is equidistributed in $\mathcal{P}$.*

As a trivial corollary, we deduce that the half-infinite 1-direction geodesic spends asymptotically the same amount of time in each one of the n cubes of the n-cube 3-manifold.

We comment here also that the proof technique of Theorem 1.1 works in any dimension and leads to analogous results.

Equidistribution leads to density. For 2-dimensional continuous flat systems, we have alternative and simpler arguments that lead to density results without the need to establish equidistribution. Unfortunately, we do not have similar techniques for 3-dimensional continuous flat systems. Perhaps this is an interesting problem to pursue.

2. More equidistribution results

A finite *polycube region* consists of a finite number of unit size cubes such that (i) any two cubes either are disjoint, or have a common face, or have a common edge, or have a common vertex; and (ii) there is face-connectivity, that any two cubes are joined by a chain of cubes such that any two consecutive members of the chain share a common face.

Given such a finite polycube region, we can identify pairs of xy-parallel square faces and pairs of xz-parallel square faces on the boundary surface by perpendicular translation. We can also 2-color all the yz-parallel square faces red and green, where red denotes barriers and green denotes gates. Here we drop the earlier and strong restriction that the 2-coloring must be repeated. For an illustration, see Figure 2.1.

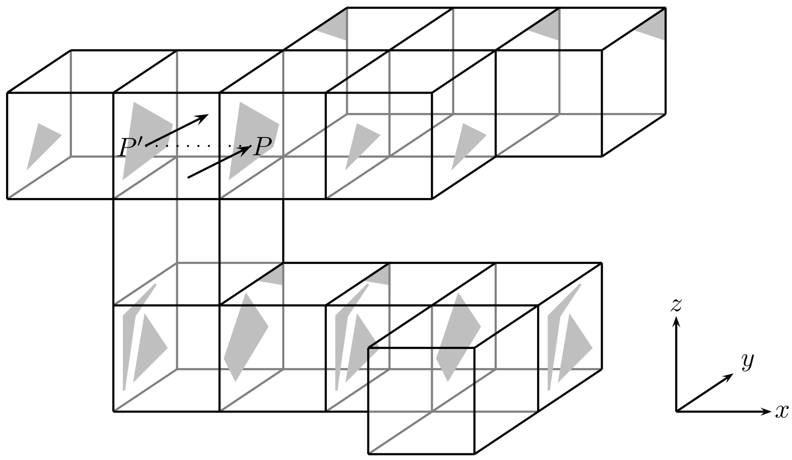

Figure 2.1: a polycube with barriers and gates on yz-parallel square faces

To obtain a flat dynamical system, we need to define 1-direction linear geodesic flow. Suppose that the direction vector is given by $\mathbf{v} = (1, \alpha_1, \alpha_2) \in \mathbb{R}^3$.

First of all, if such a geodesic hits an xy-parallel square face, then it jumps to the corresponding point on the identified xy-parallel square face and continues in the same direction, and if such a geodesic hits an xz-parallel square face, then it jumps to the corresponding point on the identified xz-parallel square face and continues in the same direction.

On the other hand, if such a geodesic hits a yz-parallel square face at a point P, then the outcome depends on the coloring of the intersection point P. If P is green, then the geodesic continues on its way in the same direction. If P is red, then we consider a directed line starting from P in the direction $(-1, 0, 0)$. This line will hit a red point P' for the first time. Then the geodesic continues from P' in the same direction, as shown in Figure 2.1.

There is the pathological case that $P' = P$, so that the geodesic continues on its way, like when the geodesic hits a green point. To avoid such situations, we deem the point P to be colored green. This is some kind of automatic recoloring, but the following rule is a little simpler.

Restriction on Red Coloring. On any line perpendicular to any given yz-parallel square face, there are either no points colored red or at least 2 distinct points colored red.

We thus obtain a finite polycube 3-manifold with barriers and gates on yz-parallel faces, and we can study 1-direction geodesic flow just described in it.

Conjecture. *Let $\mathcal{P}$ be a finite polycube 3-manifold, where each of the yz-parallel faces has arbitrary 2-coloring such that each of the red and green parts is the union of finitely many polygons, and where the green part has positive area. Suppose further that the Restriction on Red Coloring holds. Then for almost every direction and for almost every starting point, the corresponding half-infinite 1-direction geodesic is equidistributed in $\mathcal{P}$.*

We are unable to prove this general conjecture. Nevertheless, with straightforward adaptations of the ideas in the proof of Theorem 1.1, we can still establish some related non-trivial results.

As a first example of what we can establish, we return to the class of n-cube 3-manifolds. Instead of requiring perfect repetition of the 2-coloring on all the yz-parallel square faces as in Theorem 1.1, we now impose the much weaker condition of *local repetition*. More precisely, we require a small *local repetition neighborhood*, in the form of a line segment with the same local 2-coloring of red and green in the two opposite side-neighborhoods. For illustration, see Figure 2.2, where the two highlighted rectangles are in the same position within the square torus.

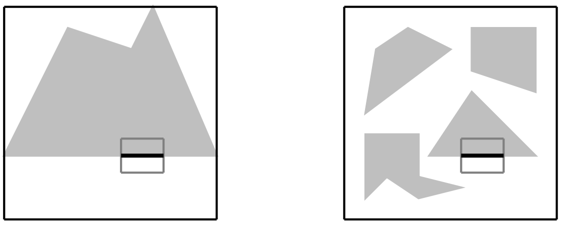

Figure 2.2: different 2-colorings with local repetition

Such a local repetition neighborhood must be present on all the yz-parallel square faces. We still require 2-colorings on each yz-parallel square face such that each of the red and green parts is the union of finitely many polygons, and where the green part has positive area. Furthermore, we also require that the Restriction on Red Coloring holds. As the 2-colorings on different yz-parallel square faces can now be different, this represents substantially more freedom for the 2-colorings. In Figure 2.3, we have a 4-cube 3-manifold with local repetition provided by the triangular red (shaded) regions at the corners of the yz-parallel square faces. The positions of the local repetition neighborhood are indicated by the short thick lines.

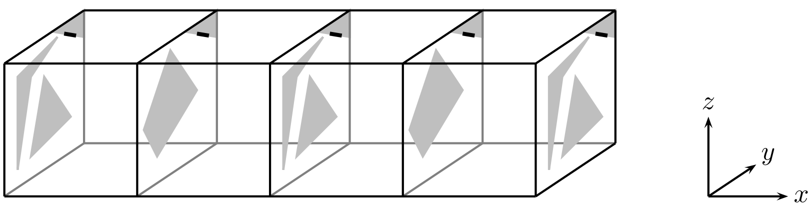

Figure 2.3: a 4-cube 3-manifold with local repetition

The result that we can prove in this more general setting is somewhat weaker.

Theorem 2.1. *Let $n \geqslant 2$ be an integer, and let $\mathcal{P}$ be any n-cube 3-manifold such that the 2-coloring on each yz-parallel square faces is such that each of the red and green parts is the union of finitely many polygons, where the green part has positive area, and the Restriction on Red Coloring holds. Suppose further that there is local repetition of the barriers on the yz-parallel square faces. Then for almost every starting point and a positive proportion of the directions $\mathbf{v} = (1, \alpha_1, \alpha_2) \in \mathbb{R}^3$ with $\mathbf{v}_0 = (\alpha_1, \alpha_2) \in [-1, 1]^2$, the corresponding half-infinite 1-direction geodesic is equidistributed in $\mathcal{P}$.*

As a further example to what we can establish, we go in a completely opposite direction. Instead of having any repetition, local or otherwise, of the 2-coloring on the yz-square faces, we consider the existence of a *private color-split neighborhood.* More precisely, in a polycube 3-manifold $\mathcal{P}$, we require one x-parallel row of cubes in $\mathcal{P}$ such that (i) it consists of at least 3 cubes of $\mathcal{P}$; (ii) the 2-coloring of one yz-parallel square face on it has a color-split neighborhood; and (iii) the same neighorhood on every other yz-parallel square face on it is monochromatic, and red for at least 2 of them.

As an illustration, Figure 2.4 shows the 2-colorings of the yz-parallel square faces of an x-parallel row of 4 cubes in a polycube 3-manifold $\mathcal{P}$. The yz-parallel square face on the left contains a color-split neighborhood. The same neighborhood is monochromatic on all the other yz-parallel square faces, and this neighborhood is red on the middle two yz-parallel square faces. Thus this color-split neighborhood on the yz-parallel square face on the left is a private color-split neighborhood.

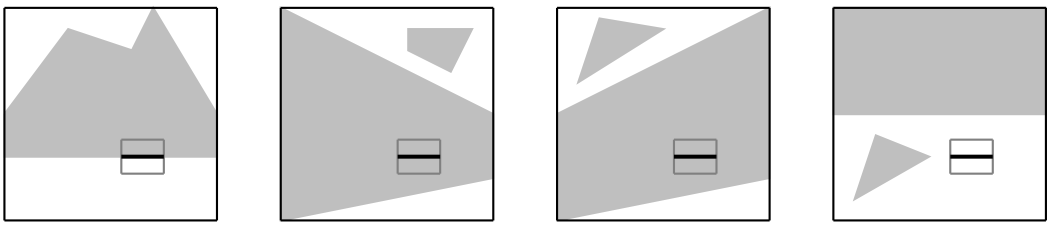

Figure 2.4: four 2-colorings with a private color-split neighborhood

Theorem 2.2. *Let $\mathcal{P}$ be a finite polycube 3-manifold such that the 2-coloring on each yz-parallel square face is such that each of the red and green parts is the union of finitely many polygons, where the green part has positive area, and the Restriction on Red Coloring holds. Suppose further that there is a private color-split neighborhood on one of the yz-parallel square faces of $\mathcal{P}$. Then for almost every starting point and almost every direction $\mathbf{v} = (1, \alpha_1, \alpha_2) \in \mathbb{R}^3$ with $\mathbf{v}_0 = (\alpha_1, \alpha_2) \in [-1, 1]^2$, the corresponding half-infinite 1-direction geodesic is equidistributed in $\mathcal{P}$.*

Figure 2.5 shows the 2-cube box with a 2-coloring on the middle yz-parallel square face such that each of the red (shaded) and green (white) parts is the union of finitely many polygons, and where the green part has positive area.

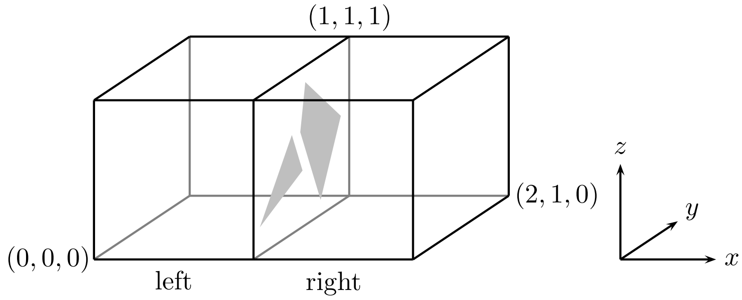

Figure 2.5: 2-cube box with barriers in the middle

Consider billiard in this 2-cube box, where in addition to the square faces on the surface of the box, there are additional barriers on the middle yz-parallel square face given by the parts colored red. As usual, we consider the ideal case of a point billiard that bounces back at any barrier, following the well-known rules of optical reflection.

We are interested in the long term bahavior of the billiard orbit. In particular, we are interested in density and uniformity. We shall show that Theorem 1.1 contributes to our understanding of such questions.

To establish equidistribution for such billiard orbits, we extend the idea of König and Szücs and apply 3-dimensional *unfolding*. This converts the billiard orbit in this 2-cube box with barriers into a 1-direction geodesic in a boundary-free flat 3-manifold. The latter system is an 8-copy construction involving 16 cubes, and results from three consecutive reflections across a plane.

The original 2-cube box with barriers in the middle is highlighted in bold in Figure 2.6. We reflect it across the plane $x = 2$, then reflect the 2-copy union across the plane $y = 1$, and finally reflect the 4-copy union across the plane $z = 1$ to obtain an 8-copy union. Thus the original $2 \times 1 \times 1$ box become a $4 \times 2 \times 2$ box with two repeated sets of barriers on the yz-parallel squares $[0, 2)^2$ on the faces $x = 1$ and $x = 3$.

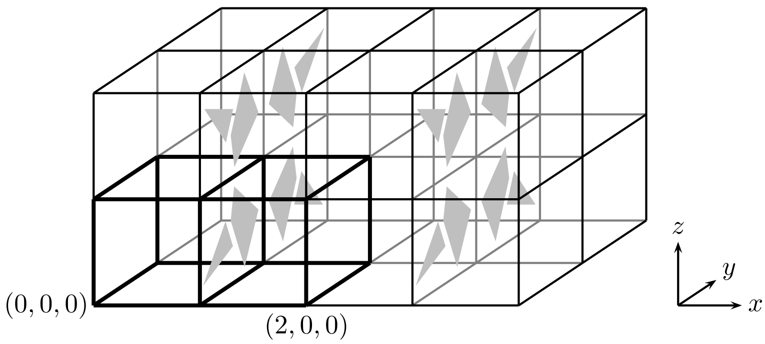

Figure 2.6: unfolding the billiard orbit in a 2-cube box
with barriers in the middle

This box has boundary, and we turn it into a boundary-free flat 3-manifold with boundary identification. First of all, the square faces on the boundary of this box are identified by perpendicular translation. Next, let L and R denote respectively the barriers on the square faces on the plane $x = 1$ and $x = 3$ respectively. The left side of L is identified with the right side of R, while the right side of L is identified with the left side of R. For convenience, we refer to this special 3-manifold as the 2*-cube-billiard* 3*-manifold.*

Clearly 1-direction geodesic flow in the 2-cube-billiard 3-manifold is an 8-fold cover of billiard flow in the 2-cube box with barriers.

If we remove the part $[0, 1) \times [0, 2) \times [0, 2)$ on the left and join it instead to the right to become $[4, 5) \times [0, 2) \times [0, 2)$, and contract the resulting 3-manifold by a factor $1/2$ in each of the three directions, we then obtain a 2-cube 3-manifold with repeated barriers that we have studied in Theorem 1.1. The following result is then a corollary of Theorem 1.1 in the special case $n = 2$.

Theorem 2.3. *Consider billiard in a special* 2*-cube box with barriers in the middle square face joining the cubes given by a* 2*-coloring such that each of the red and green parts is the union of finitely many polygons, and where the green part has positive area. Then for almost every starting point and almost every initial direction* $\mathbf{v} = (1, \alpha_1, \alpha_2) \in \mathbb{R}^3$ *with* $\mathbf{v}_0 = (\alpha_1, \alpha_2) \in [-1, 1]^2$*, the corresponding half-infinite billiard orbit is equidistributed in this special* 2*-cube box.*

Unfortunately, Theorem 1.1 does not seem to help in the case of more complicated billiards with barriers.

3. Starting the proof of Theorem 1.1

We work with the equivalent 2-dimensional discrete form of the problem.

Suppose that

$$\mathscr{R} \cup \mathscr{G} = [0,1)^2$$

denotes an arbitrary red and green 2-coloring of the unit torus $[0,1)^2$, where each of the red part $\mathscr{R}$ and the green part $\mathscr{G} = [0,1)^2 \setminus \mathscr{R}$ is the union of finitely many polygons, and where $\mathscr{G}$ has positive total area.

Let $\mathbf{v} = (1, \alpha_1, \alpha_2) \in \mathbb{R}^3$ be a Kronecker vector, and let $\mathbf{v}_0 = (\alpha_1, \alpha_2)$.

Remark. Since the collection of non-Kronecker vectors $\mathbf{v} = (1, \alpha_1, \alpha_2) \in \mathbb{R}^3$ has measure zero, we may therefore start our discussion assuming that the vector $\mathbf{v}$ is Kronecker.

For every integer $i = 0, 1, \ldots, n-1$, write

$$U_i = [i, i+1) \times [0,1), \tag{3.1}$$

and let

$$X_0 = \bigcup_{i=0}^{n-1} U_i. \tag{3.2}$$

We define an invertible transformation $T = T_{\mathbf{v}_0} : X_0 \to X_0$ as follows.

For any point $P = (i+x, y) \in U_i \subset X_0$, let

$$T(P) = \begin{cases} (i+1+\{x+\alpha_1\}, \{y+\alpha_2\}), & \text{if } (\{x+\alpha_1\}, \{y+\alpha_2\}) \in \mathscr{G}, \\ (i+\{x+\alpha_1\}, \{y+\alpha_2\}), & \text{if } (\{x+\alpha_1\}, \{y+\alpha_2\}) \in \mathscr{R}, \end{cases} \tag{3.3}$$

where $0 \leqslant \{z\} < 1$ denotes the fractional part of a real number z, so that

$$T(P) = \begin{cases} \in U_{i+1}, & \text{if } P + \mathbf{v}_0 \in \mathscr{G} \text{ modulo one}, \\ \in U_i, & \text{if } P + \mathbf{v}_0 \in \mathscr{R} \text{ modulo one}, \end{cases} \tag{3.4}$$

with the convention that $U_n = U_0 = [0,1)^2$.

Since T is basically a $\mathbf{v}_0$-shift, it preserves the 2-dimensional Lebesgue measure λ. Our goal is to prove that $T = T_{\mathbf{v}_0} : X_0 \to X_0$ is ergodic for almost every direction vector $\mathbf{v}_0 \in [-1,1]^2$. The basic idea is quite surprising, as we prove ergodicity for this non-integrable system by taking advantage of the split singularities.

We are going to use Birkhoff's well known pointwise ergodic theorem concerning measure preserving transformation twice. Since we simply apply ergodic theory, we do not expect the reader to have any serious expertise in the subject. Knowledge of Lebesgue integral and basic measure theory suffices. The theorem concerns a measure-preserving system $(X, \mathcal{A}, \mu, T)$. Here $(X, \mathcal{A}, \mu)$ is a measure space, where X is the underlying space, $\mathcal{A}$ is a σ-algebra of subsets of X and μ is a non-negative σ-additive measure on X with $\mu(X) < \infty$, while $T : X \to X$ is a measurable map which is measure-preserving, so that $T^{-1}A \in \mathcal{A}$ and $\mu(T^{-1}A) = \mu(A)$ for every $A \in \mathcal{A}$.

Let $L^1(X, \mathcal{A}, \mu)$ denote the space of measurable and integrable functions in the measure space $(X, \mathcal{A}, \mu)$. Then the general form of Birkhoff's pointwise ergodic theorem says that for every function $f \in L^1(X, \mathcal{A}, \mu)$, the limit

$$\lim_{m\to\infty} \frac{1}{m} \sum_{j=0}^{m-1} f(T^j x) = f^*(x) \tag{3.5}$$

exists for μ-almost every $x \in X$, where $f^* \in L^1(X, \mathcal{A}, \mu)$ is a T-invariant measurable function satisfying the condition

$$\int_X f \, \mathrm{d}\mu = \int_X f^* \, \mathrm{d}\mu.$$

A particularly important special case is if T is *ergodic*, when every measurable T-invariant set $A \in \mathcal{A}$ is *trivial* in the precise sense that $\mu(A) = 0$ or $\mu(A) = \mu(X)$. This is equivalent to the assertion that every measurable T-invariant function is constant μ-almost everywhere.

If T is ergodic, then (3.5) simplifies to

$$\lim_{m\to\infty} \frac{1}{m} \sum_{j=0}^{m-1} f(T^j x) = \int_X f \, \mathrm{d}\mu, \tag{3.6}$$

and the right-hand side of (3.5) is the same constant for μ-almost every $x \in X$.

The remarkable intuitive interpretation of (3.6) is that the *time average* on the left hand side is equal to the *space average* on the right hand side.

One of the earliest applications of ergodic theory in the 1930s is an extension to the Kronecker–Weyl equidistribution theorem, discussed in Section 1, from the class of all Riemann integrable functions to every individual Lebesgue integrable function. Let the integer $d \geqslant 2$ be fixed. Then the $\widetilde{T} = \widetilde{T}_{\mathbf{v}_0}$-shift acting on the unit torus $[0,1)^{d-1}$, given by $\widetilde{T}(\mathbf{x}) = \mathbf{x} + \mathbf{v}_0$ modulo one for every $\mathbf{x} \in [0,1)^{d-1}$, is ergodic if and only if $\mathbf{v}$ is a Kronecker vector. In particular, if $\mathbf{v}$ is a Kronecker vector, then (3.6) holds.

Here we focus on the particular measure-preserving system $(X_0, \mathcal{A}, \lambda, T)$, where $\mathcal{A}$ is the family of Borel sets in X_0, λ is 2-dimensional Lebesgue measure and $T = T_{\mathbf{v}_0}$. We shall establish ergodicity by contradiction.

Suppose on the contrary that T is not ergodic. Then there exists a non-trivial measurable T-invariant subset $S_0 \subset X_0$ such that $0 < \lambda(S_0) < n$. We try to derive a contradiction.

Removing possibly a set of λ-measure zero, we may assume that for every point $x \in X_0$, the point $T^j x$ is well defined for every integer $j = 1, 2, 3, \ldots$.

Lemma 3.1. *Consider the measure-preserving system $(X_0, \mathcal{A}, \lambda, T)$, where $T = T_{\mathbf{v}_0}$ and $\mathbf{v} = (1, \mathbf{v}_0)$ is a Kronecker vector. For any T-invariant subset $S_0 \subset X_0$, let the multiplicity function $\widetilde{\chi}_{S_0}$ of S_0 be defined for every point $P \in [0,1)^2$ by*

$$\widetilde{\chi}_{S_0}(P) = |\{i = 0, 1, \ldots, n-1 : P + (i, 0) \in S_0\}|.$$

Suppose further that S_0 is a proper subset of X_0, so that $S_0 \neq \emptyset$ and $S_0 \neq X_0$. Then there exists an integer $k_0 = 1, \ldots, n-1$ such that $\widetilde{\chi}_{S_0}(P) = k_0$ for almost every point $P \in [0,1)^2$, so that $\lambda(S_0) = k_0$.

Proof. Since $\mathbf{v}$ is a Kronecker vector, it follows that the $\mathbf{v}_0$-shift on the unit torus $[0,1)^2$ is ergodic. Meanwhile, it is easy to check that the multiplicity function $\widetilde{\chi}_{S_0}$ is $\widetilde{T}$-invariant. Thus Birkhoff's ergodic theorem implies that $\widetilde{\chi}_{S_0}$ is constant almost everywhere. Note that $\widetilde{\chi}_{S_0}$ is integer-valued and cannot be equal to 0 or n. This completes the proof. ∎

Given a point $z \in X_0$ and a radius $0 < r < 1/2$, let $D(z; r)$ denote the circular disk of radius r and center z. Clearly $D(z; r)$ has area πr^2. Note that $D(z; r) \subset X_0$, due to the fact that X_0 is a compact flat surface.

Since the non-trivial T-invariant subset $S_0 \subset X_0$ is measurable, it follows from Lebesgue's density theorem that for almost every $z \in S_0$,

$$\lim_{r\to 0}\frac{\lambda(S_0\cap D(z;r))}{\pi r^2}=1,$$

whereas for almost every $z \in S_0^c = X_0 \setminus S_0$,

$$\lim_{r\to 0}\frac{\lambda(S_0\cap D(z;r))}{\pi r^2}=0.$$

Let M be a large integer, and divide each of $U_0, U_1, \ldots, U_{n-1}$ into M^2 congruent squares of area $(1/M)^2$ in the standard way. We refer to these small squares as special $(1/M)$-squares. Thus there are precisely nM^2 special $(1/M)$-squares in X_0.

In view of Lebesgue's density theorem, we formulate the following lemma for the hypothetical non-trivial measurable T-invariant subset $S_0 \subset X_0$.

Lemma 3.2. *Let the real number $\varepsilon \in (0,1)$ be arbitrarily small and fixed, and let the real number $\varepsilon' > 0$ be fixed. There exists a finite threshold $m_0 = m_0(S_0;\varepsilon;\varepsilon')$ such that for every integer $M \geqslant m_0$, there exist at least $(1-\varepsilon')nM^2$ special $(1/M)$-squares Q in X_0 such that either*

$$\frac{\lambda(S_0\cap Q)}{(1/M)^2}>1-\varepsilon \quad \textit{or} \quad \frac{\lambda(S_0\cap Q)}{(1/M)^2}<\varepsilon.$$

Proof. Since S_0 is Lebesgue measurable, given any $\delta > 0$, there exists a finite set of disjoint axis-parallel rectangles such that their union V satisfies

$$\lambda(V\setminus S_0)+\lambda(S_0\setminus V)<\delta.$$

Suppose that $0 < \lambda(S_0) = \tau < n$. Then

$$\lambda(V)>\lambda(S_0)-\delta=\tau-\delta \quad\text{and}\quad \lambda(S_0^c\cap V)<\delta,$$

where $S_0^c = X_0 \setminus S_0$. Since V is a finite union of disjoint axis-parallel rectangles, there clearly exists a threshold $t_1 = t_1(V;\delta)$ such that the union V_1 of the special $(1/t)$-squares Q contained in V has measure

$$\lambda(V_1)>\lambda(V)-\delta>\tau-2\delta,$$

provided that the integer $t \geqslant t_1$. Let $\mathcal{B}$ denote the set of special $(1/t)$-squares Q in V_1 that satisfy

$$\frac{\lambda(S_0^c\cap Q)}{(1/t)^2}\geqslant\varepsilon.$$

Then, provided that $t \geqslant t_1(V;\delta)$, we have

$$\delta>\lambda(S_0^c\cap V)\geqslant\lambda(S_0^c\cap V_1)=\sum_{Q\subseteq V_1}\lambda(S_0^c\cap Q)\ \geqslant\sum_{Q\in\mathcal{B}}\lambda(S_0^c\cap Q)\geqslant\frac{\varepsilon|\mathcal{B}|}{t^2},$$

so that

$$|\mathcal{B}|\leqslant\frac{\delta t^2}{\varepsilon}=\frac{\varepsilon' t^2}{6},$$

if we choose $\delta = \varepsilon\varepsilon'/6$. Deleting the special $(1/t)$-squares $Q \in \mathcal{B}$, we see that V_1 contains at least

$$\left(\tau-2\delta-\frac{\varepsilon'}{6}\right)t^2\geqslant\left(\tau-\frac{\varepsilon'}{2}\right)t^2 \tag{3.7}$$

special $(1/t)$-squares Q such that

$$\frac{\lambda(S_0^c\cap Q)}{(1/t)^2}<\varepsilon.$$

It follows that, as long as the integer $t \geqslant t_1(V;\delta)$, the number of special $(1/t)$-squares in X_0 that satisfy

$$\frac{\lambda(S_0 \cap Q)}{(1/t)^2} > 1 - \varepsilon \tag{3.8}$$

is bounded below by (3.7). Repeating the same argument but replacing S_0 by S_0^c, we obtain another threshold $t_2 = t_2(V;\delta)$ such that, as long as the integer $t \geqslant t_2(V;\delta)$, the number of special $(1/t)$-squares in X_0 that satisfy

$$\frac{\lambda(S_0 \cap Q)}{(1/t)^2} < \varepsilon \tag{3.9}$$

is bounded below by

$$\left(n - \tau - \frac{\varepsilon'}{2}\right) t^2. \tag{3.10}$$

Combining the lower bounds (3.7) and (3.10), we see that, provided that an integer $M \geqslant \max\{t_1, t_2\}$, the number of special $(1/M)$-squares in X_0 that satisfy (3.8) or (3.9) is bounded below by $(n - \varepsilon')M^2$, and this completes the proof. □

We need two more technical lemmas which we shall establish in Section 5. The first of these results demonstrates that the overwhelming majority of arithmetic progressions are not *clustered*.

Lemma 3.3. *Let the real number $\varepsilon \in (0,1)$ be arbitrarily small and fixed. There exists a finite constant $C^* = C^*(\varepsilon)$ such that for any starting point $\mathbf{s}$ in the unit torus $[0,1)^2$ and for at least $(1-\varepsilon)$-proportion of the vectors $\mathbf{v}_0 \in [-1,1]^2$, every axis-parallel square A of side length $1/N$ in $[0,1)^2$ contains modulo one at most C^* elements of the arithmetic progression*

$$\mathbf{s} + j\mathbf{v}_0, \quad j = 0, 1, \ldots, N^2 - 1,$$

of N^2 terms. In particular, we can take $C^* = 1 + 4/\varepsilon$.

To state the next technical lemma, we first need a few definitions.

Let $A \subset [0,1)^2$ be an arbitrary axis-parallel square of side length $1/N$. Consider the N^2 $\mathbf{v}_0$-shift images

$$A + j\mathbf{v}_0, \quad j = 0, 1, \ldots, N^2 - 1, \tag{3.11}$$

modulo one. Recall the 2-coloring $\mathscr{R} \cup \mathscr{G} = [0,1)^2$. We say that $A + j\mathbf{v}_0$ is a red member of the sequence (3.11) if $A + j\mathbf{v}_0 \subset \mathscr{R}$ modulo one and a green member of the sequence (3.11) if $A + j\mathbf{v}_0 \subset \mathscr{G}$ modulo one. Furthermore, we say that $A + j\mathbf{v}_0$ is a color-split member of the sequence (3.11) if

$$(A + j\mathbf{v}_0) \cap \mathscr{R} \neq \emptyset \quad \text{and} \quad (A + j\mathbf{v}_0) \cap \mathscr{G} \neq \emptyset$$

modulo one.

We say that a set $\{A + j\mathbf{v}_0 : j \in J\}$, where J is a subset of consecutive integers in $\{0, 1, \ldots, N^2 - 1\}$, is called split-free if every member is either a red member or a green member of (3.11). Furthermore, we say that the set $\{A + j\mathbf{v}_0 : j \in J\}$ is a split-free chain if it is split-free and not contained in a bigger split-free set.

Thus the sequence (3.11) decomposes into a subsequence of color-split members, with any two consecutive members of this subsequence possibly separated by a split-free chain in between.

Lemma 3.4. *Let the real number $\varepsilon \in (0,1)$ be arbitrarily small and fixed. There exists a positive absolute constant C_1 such that for any axis-parallel square A of side length $1/N$ and for at least $(1-\varepsilon)$-proportion of the vectors $\mathbf{v}_0 \in [-1,1]^2$, the sequence* (3.11) *modulo one has at most εN split-free chains with length at most $C_1 \varepsilon^2 N$.*

To formulate the last lemma in this section, we need to define the concept of *double-step with color-split*. There are two versions. As before, let $\widetilde{T}$ denote the projection of $T = T_{\mathbf{v}_0}$ to the unit torus $[0,1)^2$. Let A denote an axis-parallel square of side length $1/N$ in U_i, where $i = 0, 1, \ldots, n-1$.

Double-Step with Color-Split and Green Arrival. Here $\widetilde{T}(A)$ exhibits color-split and $\widetilde{T}^2(A) \subset \mathscr{G}$. More precisely, we extend the 2-coloring $\mathscr{R} \cup \mathscr{G} = [0,1)^2$ to all the faces U_i, $i = 0, 1, \ldots, n-1$. Let $A \subset U_i \subset X_0$, where $i = 0, 1, \ldots, n-3$. Then $T(A)$ exhibits color-split, so that

$$T(A) \cap \mathscr{G} \neq \emptyset \quad \text{and} \quad T(A) \cap \mathscr{R} \neq \emptyset.$$

For the green part, we have

$$T(A) \cap \mathscr{G} \subset U_{i+1} \quad \text{and} \quad T(T(A) \cap \mathscr{G}) \subset U_{i+2}.$$

For the red part, we have

$$T(A) \cap \mathscr{R} \subset U_i \quad \text{and} \quad T(T(A) \cap \mathscr{R}) \subset U_{i+1}.$$

We can therefore formally describe this process by

$$U_i \mapsto_{\mathscr{G}} (U_{i+2}, U_{i+1}), \quad i = 0, 1, \ldots, n-3. \tag{3.12}$$

Similarly, we can also write

$$U_{n-2} \mapsto_{\mathscr{G}} (U_0, U_{n-1}) \quad \text{and} \quad U_{n-1} \mapsto_{\mathscr{G}} (U_1, U_0). \tag{3.13}$$

Double-Step with Color-Split and Red Arrival. Here $\widetilde{T}(A)$ exhibits color-split and $\widetilde{T}^2(A) \subset \mathscr{R}$. More precisely, we extend the 2-coloring $\mathscr{R} \cup \mathscr{G} = [0,1)^2$ to all the faces U_i, $i = 0, 1, \ldots, n-1$. Let $A \subset U_i \subset X_0$, where $i = 0, 1, \ldots, n-2$. Then $T(A)$ exhibits color-split, so that

$$T(A) \cap \mathscr{G} \neq \emptyset \quad \text{and} \quad T(A) \cap \mathscr{R} \neq \emptyset.$$

For the green part, we have

$$T(A) \cap \mathscr{G} \subset U_{i+1} \quad \text{and} \quad T(T(A) \cap \mathscr{G}) \subset U_{i+1}.$$

For the red part, we have

$$T(A) \cap \mathscr{R} \subset U_i \quad \text{and} \quad T(T(A) \cap \mathscr{R}) \subset U_i.$$

We can therefore formally describe this process by

$$U_i \mapsto_{\mathscr{R}} (U_{i+1}, U_i), \quad i = 0, 1, \ldots, n-2. \tag{3.14}$$

Similarly, we can also write

$$U_{n-1} \mapsto_{\mathscr{R}} (U_0, U_{n-1}). \tag{3.15}$$

The last ingredient in the proof of Theorem 1.1 is the following lemma. The proof is very simple.

Lemma 3.5 (split lemma). *Let*

$$\mathcal{X} = \{U_i : i = 0, 1, \ldots, n-1\},$$

with the convention that $U_n = U_0$ and $U_{n+1} = U_1$. For any subset $\mathcal{Z} \subset \mathcal{X}$, write

$$\mathcal{Z} = \{U_i : i \in I_{\mathcal{Z}}\}, \quad I_{\mathcal{Z}} \subset \{0, 1, \ldots, n-1\}.$$

(i) *For any subset $\mathcal{Z} \subset \mathcal{X}$, let $f_0(\mathcal{Z}) \subset \mathcal{X}$ be defined by*

$$f_0(\mathcal{Z}) = \bigcup_{i \in I_{\mathcal{Z}}} f_0(U_i),$$

where

$$f_0(U_i) = \{U_{i+1}, U_{i+2}\}, \quad i = 0, 1, \ldots, n-1,$$

using the rules (3.12) *and* (3.13) *of double-step with color-split and green arrival. Suppose that* $\mathcal{Z} \neq \emptyset$ *and* $\mathcal{Z} \neq \mathcal{X}$. *Then* $|f_0(\mathcal{Z})| > |\mathcal{Z}|$.

(ii) *For any subset* $\mathcal{Z} \subset \mathcal{X}$, *let* $g_0(\mathcal{Z}) \subset \mathcal{X}$ *be defined by*

$$g_0(\mathcal{Z}) = \bigcup_{i \in I_{\mathcal{Z}}} g_0(U_i),$$

where

$$g_0(U_i) = \{U_i, U_{i+1}\}, \quad i = 0, 1, \ldots, n-1,$$

using the rules (3.14) *and* (3.15) *of double-step with color-split and red arrival. Suppose that* $\mathcal{Z} \neq \emptyset$ *and* $\mathcal{Z} \neq \mathcal{X}$. *Then* $|g_0(\mathcal{Z})| > |\mathcal{Z}|$.

Proof. We shall only prove part (i), as the proof of part (ii) is similar. Note that

$$U_i \to U_{i+1} \quad \text{and} \quad U_i \to U_{i+2}$$

are respectively a 1-shift and a 2-shift on an n-cycle. As each of these shifts preserves cardinality, it is clear that $|f_0(\mathcal{Z})| \geqslant |\mathcal{Z}|$ always. Suppose that $|f_0(\mathcal{Z})| = |\mathcal{Z}|$ for some subset $\mathcal{Z} \subset \mathcal{X}$ such that $\emptyset \neq \mathcal{Z} \neq \mathcal{X}$. Then the 1-shift image and 2-shift image of $\mathcal{Z}$ coincide, so that $\mathcal{Z}$ is 1-shift invariant. Since $\mathcal{Z} \neq \emptyset$, there exists some $i_0 = 0, 1, \ldots, n-1$ such that $U_{i_0} \in \mathcal{Z}$. Then $U_{i_0+1} \in \mathcal{Z}$, $U_{i_0+2} \in \mathcal{Z}$, and so on, so that $\mathcal{Z} = \mathcal{X}$, clearly a contradiction. □

4. Completing the proof of Theorem 1.1

As discussed just before Lemma 3.1 in Section 3, we focus on the particular measure-preserving system $(X_0, \mathcal{A}, \lambda, T)$, where the set X_0 is given by (3.1) and (3.2), $\mathcal{A}$ is the family of Borel sets in X_0, λ is 2-dimensional Lebesgue measure and the transformation $T = T_{\mathbf{v}_0}$ is defined by (3.3) and (3.4).

We shall first establish ergodicity of T by contradiction.

Suppose on the contrary that T is not ergodic. Then there exists a measurable T-invariant subset $S_0 \subset X_0$ such that $0 < \lambda(S_0) < n$. We proceed in steps.

Step 1. By Lemma 3.1, there exists an integer $k_0 = 1, \ldots, n-1$ such that the multiplicity function $\widetilde{\chi}_{S_0}(P) = k_0$ for almost every point $P \in [0,1)^2$.

Step 2. This step is an application of Lemma 3.2. Assume that N is a large even integer. Let $\mathcal{F}(N/2)$ denote the standard decomposition of the unit torus $[0,1)^2$ into $(N/2)^2$ axis-parallel congruent small squares of common side length $2/N$ such that the origin $(0,0)$ is the vertex of a small square. For $\boldsymbol{\delta} = (\delta_1, \delta_2) \in \{0,1\}^2$, let $\mathcal{F}_{\boldsymbol{\delta}}(N/2)$ denote the translation of $\mathcal{F}(N/2)$ modulo one such that the vertex $(0,0)$ moves to $(\delta_1/N, \delta_2/N)$. We refer to the small squares in the four partitions $\mathcal{F}_{\boldsymbol{\delta}}(N/2)$, $\boldsymbol{\delta} \in \{0,1\}^2$, as basic $(2/N)$-squares. It is not difficult to see that any axis-parallel square B of side length $1/N$ in the unit torus is contained in a basic $(2/N)$-square.

Next, we extend the families $\mathcal{F}_{\boldsymbol{\delta}}(N/2)$, $\boldsymbol{\delta} \in \{0,1\}^2$, in the unit torus $[0,1)^2 = U_0$ to every unit torus $U_i = [i, i+1) \times [0,1)$, $i = 0, 1, \ldots, n-1$, by translation by the vector $(i, 0)$. In other words.

$$\mathcal{F}_{\boldsymbol{\delta}}(N/2; i) = \mathcal{F}_{\boldsymbol{\delta}}(N/2) + (i, 0), \quad \boldsymbol{\delta} \in \{0,1\}^2, \quad i = 0, 1, \ldots, n-1.$$

For every $\boldsymbol{\delta} \in \{0,1\}^2$, Lemma 3.2 with $M = N/2$ then gives the following.

Let the real number $\varepsilon \in (0,1)$ be arbitrarily small and fixed, and let the real number $\varepsilon' > 0$ be fixed. There exists a finite threshold $m_0 = m_0(S_0; \varepsilon; \varepsilon')$ such that for every $\boldsymbol{\delta} \in \{0,1\}^2$ and every even integer $N \geqslant m_0$, there exist at least $(1 - \varepsilon')n(N/2)^2$ basic $(2/N)$-squares

$$Q \in \bigcup_{i=0}^{n-1} \mathcal{F}_{\boldsymbol{\delta}}(N/2; i) \tag{4.1}$$

such that either

$$\frac{\lambda(S_0\cap Q)}{(2/N)^2}>1-\varepsilon \quad\text{or}\quad \frac{\lambda(S_0\cap Q)}{(2/N)^2}<\varepsilon. \tag{4.2}$$

Step 3. This step concerns an averaging argument. Let $A\subset[0,1)^2$ be an arbitrary axis-parallel square of side length $1/N$ in the unit torus $[0,1)^2$. We shall show that there exists a *translation* $\mathbf{t}^*\in[0,1)^2$ such that the set $A^*=A+\mathbf{t}^*\subset[0,1)^2$ modulo one satisfies the following property. There exists a positive constant C_2, depending only on the red and green 2-coloring, such that there are at least C_2N members of the sequence

$$A^*+j\mathbf{v}_0,\quad j=0,1,\ldots,N^2-1, \tag{4.3}$$

modulo one that exhibit *substantial* color-split, in the sense that

$$\lambda((A^*+j\mathbf{v}_0)\cap\mathscr{R})\geqslant\frac{1}{4N^2}\quad\text{and}\quad\lambda((A^*+j\mathbf{v}_0)\cap\mathscr{G})\geqslant\frac{1}{4N^2}. \tag{4.4}$$

To prove this, recall that the red and green 2-coloring of the unit torus $[0,1)^2$ is determined by finitely many polygons. Hence there exists a positive constant C_2 and a polygonal subset $T\subset[0,1)^2$ with area $\lambda(T)\geqslant C_2/N$ such that for every $\mathbf{t}\in T$,

$$\lambda((A+\mathbf{t})\cap\mathscr{R})\geqslant\frac{1}{4N^2}\quad\text{and}\quad\lambda((A+\mathbf{t})\cap\mathscr{G})\geqslant\frac{1}{4N^2}. \tag{4.5}$$

Consider the translated sets

$$T_j=T-j\mathbf{v}_0,\quad j=0,1,\ldots,N^2-1,$$

modulo one, and consider the function

$$F=\sum_{j=0}^{N^2-1}\chi_{T_j} \tag{4.6}$$

which is the sum of the characteristic functions χ_{T_j}, $j=0,1,\ldots,N^2-1$. Suppose that $\mathbf{t}^*\in[0,1)^2$ is a point where F attains its maximum. Then

$$F(\mathbf{t}^*)\geqslant\int_0^1\!\!\int_0^1F(y,z)\,\mathrm{d}y\,\mathrm{d}z=\sum_{j=0}^{N^2-1}\int_0^1\!\!\int_0^1\chi_{T_j}(y,z)\,\mathrm{d}y\,\mathrm{d}z\geqslant C_2N. \tag{4.7}$$

Note from (4.6) and (4.7) that there are at least C_2N values of $j=0,1,\ldots,N^2-1$ such that $\mathbf{t}^*\in T_j$. For each of these values of j, we have $\mathbf{t}=\mathbf{t}^*+j\mathbf{v}_0\in T$, so that (4.5) and hence (4.4) is satisfied.

Step 4. Lemma 3.4 gives, for any axis-parallel square A of side length $1/N$ and for at least $(1-\varepsilon)$-proportion of the vectors $\mathbf{v}_0\in[-1,1]^2$, an upper bound for the number of short split-free chains in the sequence (4.3), and Step 3 gives a lower bound for the number of elements of (4.3) modulo one that exhibit substantial color-split. Removing all the short split-free chains from the sequence (4.3), we obtain a substantial number of split-free chains that are not short.

More precisely, let the real number $\varepsilon\in(0,1)$ be arbitrarily small and fixed. Then for any axis-parallel square A of side length $1/N$ and for at least $(1-\varepsilon)$-proportion of the vectors $\mathbf{v}_0\in[-1,1]^2$, there exist at least $(C_2-2\varepsilon)N/2$ disjoint consecutive pairs of split-free chains that are separated by a substantial color-split member of the sequence (4.3), and where both split-free chains have length greater than $C_1\varepsilon^2N$. Assuming that $0<\varepsilon\leqslant C_2/4$, we obtain at least $C_2N/4$ such disjoint consecutive pairs of *long* split-free chains that are separated by a substantial color-split member of the sequence (4.3).

Step 5. In this step, a combination of Step 2 and Lemma 3.3 shows that a majority of color-split chains contain members which satisfy an *ε-nearly zero-one law.* More precisely, let the real number $\varepsilon \in (0,1)$ be arbitrarily small and fixed. Then for any axis-parallel square A of side length $1/N$ and for at least $(1-\varepsilon)$-proportion of the vectors $\mathbf{v}_0 \in [-1,1]^2$, each of the disjoint consecutive pairs of split-free chains from the sequence (4.3) in Step 4, of length greater than $C_1\varepsilon^2 N$ and separated by a substantial color-split member, contains two members

$$A^* + j_1\mathbf{v}_0 \quad \text{and} \quad A^* + j_2\mathbf{v}_0 \tag{4.8}$$

in the two constituent long split-free chains for which there are basic $(2/N)$-squares Q_1 and Q_2 such that

$$A^* + j_\ell\mathbf{v}_0 \subset Q_\ell \in \bigcup_{\boldsymbol{\delta}\in\{0,1\}^2} \mathcal{F}_{\boldsymbol{\delta}}(N/2), \quad \ell = 1,2,$$

modulo one, and such that for $\ell = 1,2$ and every $i = 0,1,\ldots,n-1$, either

$$\frac{\lambda(S_0 \cap Q_\ell(i))}{(2/N)^2} > 1-\varepsilon \quad \text{or} \quad \frac{\lambda(S_0 \cap Q_\ell(i))}{(2/N)^2} < \varepsilon, \tag{4.9}$$

where $Q_\ell(i) = Q_\ell + (i,0) \subset U_i$ is a translated copy.

We shall prove this by contradiction.

By Step 4 with $\varepsilon/2$ replacing ε, for any axis-parallel square A of side length $1/N$ and for at least $(1-\varepsilon/2)$-proportion of the vectors $\mathbf{v}_0 \in [-1,1]^2$, there are at least $C_2N/4$ disjoint consecutive pairs of split-free chains of length greater than $C_1\varepsilon^2 N$ from the sequence (4.3). By hypothesis, for each such pair of split-free chains, each of the at least $C_1\varepsilon^2N - 2$ split-free members of at least one of the two split-free chains fails (4.9) for some $i = 0,1,\ldots,n-1$.

By Lemma 3.3 with $\varepsilon/2$ replacing ε, the *clustering constant* $C^*(\varepsilon/2) = 1 + 8/\varepsilon$ can be used here. Thus for at least $(1-\varepsilon)$-proportion of the vectors $\mathbf{v}_0 \in [-1,1]^2$, we obtain at least

$$\frac{C_2N}{4}(C_1\varepsilon^2N - 2)\cdot\frac{\varepsilon}{8+\varepsilon} \geqslant C_3\varepsilon^3N^2$$

distinct basic $(2/N)$-squares

$$Q \in \bigcup_{\boldsymbol{\delta}\in\{0,1\}} \bigcup_{i=0}^{n-1} \mathcal{F}_{\boldsymbol{\delta}}(N/2;i) \tag{4.10}$$

such that (4.9) fails.

On the other hand, it follows from Step 2, in particular (4.1) and (4.2), that there are at most $4\varepsilon' n(N/2)^2$ basic $(2/N)$-squares (4.10) such that (4.9) fails. Choosing $\varepsilon' > 0$ to satisfy $C_3\varepsilon^3 > n\varepsilon'$ now leads to a contradiction.

Step 6. In this step, we make statements analogous to (4.9) but concerning the $(1/N)$-squares (4.8) instead of the basic $(2/N)$-squares Q_ℓ that contain them.

Let the real number $\varepsilon \in (0,1)$ be arbitrarily small and fixed. Then for at least $(1-\varepsilon)$-proportion of the vectors $\mathbf{v}_0 \in [-1,1]^2$, we have the following.

For $\ell = 1,2$ and $i = 0,1,\ldots,n-1$, write

$$A^*(j_\ell;i) = (A^* + j_\ell\mathbf{v}_0) + (i,0) \subset [i,i+1)\times[0,1) = U_i$$

modulo one. It then follows from Step 5, and (4.9) in particular, that for $\ell = 1,2$ and every $i = 0,1,\ldots,n-1$, either

$$\frac{\lambda(S_0 \cap A^*(j_\ell;i))}{(1/N)^2} > 1-4\varepsilon \tag{4.11}$$

or

$$\frac{\lambda(S_0 \cap A^*(j_\ell; i))}{(1/N)^2} < 4\varepsilon. \tag{4.12}$$

Since the multiplicity function $\widetilde{\chi}_{S_0}$ is equal to an integer $k_0 = 1, \ldots, n-1$ almost everywhere, it follows that if ε is sufficiently small, then for $\ell = 1, 2$, there are precisely k_0 distinct values of $i = 0, 1, \ldots, n-1$ such that the inequality (4.11) holds and precisely $n - k_0$ distinct values of $i = 0, 1, \ldots, n-1$ such that the inequality (4.12) holds.

Step 7. This is the crucial step where we use Lemma 3.5 and take advantage of the split singularities. Let the real number $\varepsilon \in (0, 1)$ be arbitrarily small and fixed. Then for at least $(1 - \varepsilon)$-proportion of the vectors $\mathbf{v}_0 \in [-1, 1]^2$, we have the following.

Assume that $j_1 < j_2$, and let $A^* + j^*\mathbf{v}_0$ denote the color-split member of the sequence (4.3) that separates the two consecutive chains that contain (4.8). Then clearly $j_1 < j^* < j_2$. We shall focus on the effect of the transition from $j^* - 1$ to $j^* + 1$.

Suppose that the integer j satisfies $j_1 \leqslant j \leqslant j^* - 1$. Then $A^* + j\mathbf{v}_0$ is inside a split-free chain that also contains $A^* + j_1\mathbf{v}_0$. Since $T = T_{\mathbf{v}_0}$ is measure-preserving and is equal to the $\mathbf{v}_0$-shift on the unit torus $[0, 1)^2$ modulo one, it follows that there are precisely k_0 distinct values of $i = 0, 1, \ldots, n-1$ such that the inequality (4.11) holds with j_ℓ replaced by j and precisely $n - k_0$ distinct values of $i = 0, 1, \ldots, n-1$ such that the inequality (4.12) holds with j_ℓ replaced by j. In particular, there are precisely k_0 distinct values of $i = 0, 1, \ldots, n-1$ such that the inequality

$$\frac{\lambda(S_0 \cap A^*(j^* - 1; i))}{(1/N)^2} > 1 - 4\varepsilon \tag{4.13}$$

holds.

The transition from $j^* - 1$ to $j^* + 1$ is what we call the double-step with color-split, as $A^* + j^*\mathbf{v}_0$ exhibits substantial color-split in the sense of (4.4). Applying Lemma 3.5 with a subset $\mathcal{Z}$, where $I_{\mathcal{Z}}$ contains precisely the k_0 distinct values of $i = 0, 1, \ldots, n-1$ such that the inequality (4.13) holds, we deduce in both cases of green or red arrival that there is a *split-increase*, and there are at least $k_0 + 1$ distinct values of $i = 0, 1, \ldots, n-1$ such that the inequality

$$\frac{\lambda(S_0 \cap A^*(j^* + 1; i))}{(1/N)^2} > \frac{1}{4} - 4\varepsilon$$

holds. On the other hand, note that $A^* + (j^* + 1)\mathbf{v}_0$ is inside a split-free chain that also contains $A^* + j_2\mathbf{v}_0$. The measure-preserving property of the $\mathbf{v}_0$-shift on the unit torus $[0, 1)^2$ modulo one now implies that there are at least $k_0 + 1$ distinct values of $i = 0, 1, \ldots, n-1$ such that the inequality

$$\frac{\lambda(S_0 \cap A^*(j_2; i))}{(1/N)^2} > \frac{1}{4} - 4\varepsilon \tag{4.14}$$

holds. But then the last statement in Step 6 asserts that there are precisely $n - k_0$ distinct values of $i = 0, 1, \ldots, n-1$ such that the inequality

$$\frac{\lambda(S_0 \cap A^*(j_2; i))}{(1/N)^2} < 4\varepsilon \tag{4.15}$$

holds. If $\varepsilon \leqslant 1/32$, then (4.14) and (4.15) contradict each other.

This contradiction proves that the measure-preserving transformation $T = T_{\mathbf{v}_0}$ defined by (3.3) and (3.4) is ergodic for at least $(1 - \varepsilon)$-proportion of the vectors $\mathbf{v}_0 \in [-1, 1]^2$. As $\varepsilon > 0$ is arbitrarily small and N can be taken arbitrarily large, it follows that $T = T_{\mathbf{v}_0}$ is ergodic for almost all vectors $\mathbf{v}_0 \in [-1, 1]^2$.

Remark. Switching to the equivalent 3-dimensional continuous model, we eatsblish ergodicity for almost all vectors of the form

$$\mathbf{v} = (1, \mathbf{v}_0) \in \mathbb{R}^3, \quad \mathbf{v}_0 \in [-1,1]^2. \tag{4.16}$$

We traditionally identify the set of all 3-dimensional directions with the points on the unit sphere $x^2 + y^2 + z^2 = 1$. Using this identification, (4.16) defines a subset of the unit sphere with positive surface area. A straightforward extension of our argument above gives ergodicity for almost all 3-dimensional direction vectors on the whole unit sphere.

Finally, to complete the proof of Theorem 1.1, note that in view of ergodicity, an application of Birkhoff's theorem (3.6) implies equidistribution for almost every half-infinite orbit. However, to be precise, we need to clarify the difference between (3.6) and the concept of Weyl equidistribution in Theorem 1.1.

The assertion (3.6) applies for every individual Lebesgue-measurable test set and holds for almost every starting point, depending on the given test set. To prove ordinary, or Weyl, equidistribution in Theorem 1.1, it suffices to apply (3.6) to the family of all polygons in X_0 for which all the vertices have rational coordinates in the 2-dimensional discrete model. This family of test sets is countable, and since a countable union of sets of measure zero has measure zero, (3.6) implies the desired equidistribution. This completes the proof of Theorem 1.1, apart from the proofs of Lemmas 3.3 and 3.4.

5. Proof of Lemmas 3.3 and 3.4

In this section, $\mathbf{v} = (1, \alpha_1, \alpha_2) \in \mathbb{R}^3$ denotes a Kronecker vector, and $\mathbf{v}_0 = (\alpha_1, \alpha_2)$.

Proof of Lemma 3.3. For any integer $k = 1, \ldots, N^2$, write

$$\Omega(N;k) = \left\{ \mathbf{v}_0 = (\alpha_1, \alpha_2) \in [-1,1]^2 : \|k\alpha_1\| < \frac{1}{N} \text{ and } \|k\alpha_2\| < \frac{1}{N} \right\}. \tag{5.1}$$

It is not difficult to see that

$$\Omega(N;k) \cap [0,1)^2 = \left(\left[0, \frac{1}{kN}\right) \cup \bigcup_{j=1}^{k-1} \left(\frac{j}{k} - \frac{1}{kN}, \frac{j}{k} + \frac{1}{kN} \right) \cup \left(1 - \frac{1}{kN}, 1\right) \right)^2,$$

and also that

$$\lambda(\Omega(N;k)) = 4\lambda(\Omega(N;k) \cap [0,1)^2) = 4\left(\frac{2}{N}\right)^2 = \frac{16}{N^2}. \tag{5.2}$$

For every $(\alpha_1, \alpha_2) \in [-1,1]^2$, consider the counting function

$$\omega_N(\alpha_1, \alpha_2) = |\{k = 1, \ldots, N^2 : (\alpha_1, \alpha_2) \in \Omega(N;k)\}|. \tag{5.3}$$

Combining (5.1)–(5.3), we see that

$$\int_{-1}^{1} \int_{-1}^{1} \omega_N(\alpha_1, \alpha_2) \, \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 = \sum_{k=1}^{N^2} \lambda(\Omega(N;k)) = 16.$$

Given any $\varepsilon > 0$, write

$$\Omega(\varepsilon) = \left\{ (\alpha_1, \alpha_2) \in [-1,1]^2 : \omega_N(\alpha_1, \alpha_2) \geqslant \frac{4}{\varepsilon} \right\}.$$

Since the function $\omega_N(\alpha_1, \alpha_2)$ is non-negative, we clearly have

$$16 = \int_{-1}^{1} \int_{-1}^{1} \omega_N(\alpha_1, \alpha_2) \, \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 \geqslant \frac{4}{\varepsilon} \lambda(\Omega(\varepsilon)),$$

so that $\lambda(\Omega(\varepsilon)) \leqslant 4\varepsilon$. It follows that

$$\lambda([-1,1]^2 \setminus \Omega(\varepsilon)) > 4(1-\varepsilon),$$

so that the collection of vectors

$$\mathbf{v}_0 = (\alpha_1, \alpha_2) \in [-1,1]^2 \setminus \Omega(\varepsilon) \tag{5.4}$$

represents at least $(1-\varepsilon)$-proportion of the set $[-1,1]^2$.

We shall show that the lemma holds with the choice

$$C^* = C^*(\varepsilon) = 1 + \frac{4}{\varepsilon}.$$

Suppose on the contrary that some axis-parallel square Q with side length $1/N$ contains modulo one more than $1 + 4/\varepsilon$ elements of some sequence

$$\mathbf{s} + j\mathbf{v}_0, \quad j = 0, 1, \ldots, N^2 - 1.$$

In other words, suppose that there exists a subset $J \subset \{0, 1, \ldots, N^2 - 1\}$ with $|J| > 1 + 4/\varepsilon$ such that

$$\mathbf{s} + j\mathbf{v}_0 \in Q, \quad j \in J.$$

Let j_0 be the smallest element of J, and let

$$J^* = \{k = j - j_0 : j \in J \setminus \{j_0\}\}.$$

It is not difficult to see that

$$\{k\mathbf{v}_0 : k \in J^*\} \subset \{k = 1, \ldots, N^2 : (\alpha_1, \alpha_2) \in \Omega(N; k)\},$$

and so

$$\omega_N(\mathbf{v}_0) \geqslant |\{k\mathbf{v}_0 : k \in J^*\}| = |J^*| > \frac{4}{\varepsilon}.$$

This implies that $\mathbf{v}_0 \in \Omega(\varepsilon)$, contradicting (5.4). The lemma now follows. ☐

Proof of Lemma 3.4. Throughout this proof, the parameters $c_1, c_2, c_3, \ldots$ represent positive absolute constants.

We tile $\mathbb{R}^3$ with congruent copies of the $n \times 1 \times 1$ box, and extend the red and green 2-coloring $\mathscr{R} \cup \mathscr{G} = [0,1)^2$ to all yz-parallel unit squares arising from the lattice $\mathbb{Z}^3$. We can then visualize a half-infinite 1-direction geodesic in the n-cube 3-manifold as a half-infinite straight line in $\mathbb{R}^3$.

Let A be any axis parallel square on some yz-parallel unit square. In this model, the sequence (3.11) becomes

$$A + j\mathbf{v}, \quad j = 0, 1, \ldots, N^2 - 1. \tag{5.5}$$

We can describe a typical split-free chain in (5.5) as follows. There exist integers j_1 and j_2 satisfying $0 \leqslant j_1 < j_2 < N^2$ such that each member of the sequence

$$A + j\mathbf{v}, \quad j_1 < j < j_2, \tag{5.6}$$

is monochromatic, while $A + j_1\mathbf{v}$ and $A + j_2\mathbf{v}$ at either end are color-split.

Let $\eta = \eta(\varepsilon) > 0$, to be fixed later. We shall say that the split-free chain (5.6) is *short* if $|j_2 - j_1| \leqslant \eta N$. We want to detect *bad* directions $\mathbf{v} = (1, \mathbf{v}_0)$, where $\mathbf{v}_0 \in [-1,1]^2$, that give rise to short split-free chains. Here a direction in 3-space corresponds to a point on the unit sphere $x^2 + y^2 + z^2 = 1$ with surface area 4π, and a bad direction is given by a vector that goes from any point on $A + j_1\mathbf{v}$ to any point on $A + j_2\mathbf{v}$ at either end of a short split-free chain. Clearly the length of any such vector is at most $\sqrt{3}\eta N$, and we assume that $\sqrt{3}\eta N > 1$. Let $m > 0$ be an integer such that

$$2^{m-1} < \sqrt{3}\eta N \leqslant 2^m. \tag{5.7}$$

We remark that, for convenience, we represent the length of the split-free chain as $|j_2 - j_1|$, although the precise value is $|j_2 - j_1| - 1$.

Our basic idea is straightforward. Suppose that Q is a color-split $(1/N)$-square on some yz-parallel unit square arising from the lattice $\mathbb{Z}^3$. Consider some other color-split $(1/N)$-square Q^* on some other yz-parallel unit square arising from the lattice $\mathbb{Z}^3$, where some point in Q^* can be reached from some point in Q via a vector of length at most 2^m. Then these two color-split $(1/N)$-squares Q and Q^* determine a set of bad directions. Theoretically, we can consider all color-split $(1/N)$-squares Q on some fixed yz-parallel unit square arising from the lattice $\mathbb{Z}^3$, determine all possible color-split $(1/N)$-squares Q^* on other yz-parallel unit squares that can be reached from one of the $(1/N)$-squares Q via a vector of length at most 2^m, and collect together all the bad directions. Carrying this out, however, is impossible without more care.

Clearly every yz-parallel unit square arising from the lattice $\mathbb{Z}^3$ has N^2 special $(1/N)$-squares by dividing the unit square into congruent squares with side length $1/N$ in the standard way. Since the red and green 2-coloring is defined by a finite set of polygons, there are at most c_1N special $(1/N)$-squares that exhibit color-split in its interior or on part of its boundary. We shall refer to these as color-split special $(1/N)$-squares.

It is not difficult to see that every color-split $(1/N)$-square intersects a color-split special $(1/N)$-square and is therefore contained in the 3×3 neighborhood of a color split special $(1/N)$-square. Note here that part of this neignhorhood may fall into a neighboring yz-parallel unit square arising from the lattice $\mathbb{Z}^3$. Hence it suffices to focus on the 3×3 neighborhoods of color-split special $(1/N)$-squares. We shall say that a $(1/N)$-square is *exceptional* if it is either a color-split special $(1/N)$-square or a special $(1/N)$-square that forms part of the 3×3 neighborhood of a color-split special $(1/N)$-square.

We shall consider the set of bad direction vectors of length between 2^{r-1} and 2^r, where the positive integer $r\leqslant m$. Without loss of generality, we start from a fixed yz-parallel unit square arising from the lattice $\mathbb{Z}^3$. We take a number of steps.

(i) There are at most $9c_1N$ exceptional $(1/N)$-squares on this fixed yz-parallel unit square.

(ii) Start with an exceptional $(1/N)$-square Q on this fixed yz-parallel unit square. The number of yz-parallel unit squares arising from the lattice $\mathbb{Z}^3$ that can be reached from a point of Q via a vector of length between 2^{r-1} and 2^r is at most $c_2(2^r)^3$.

(iii) There are at most $9c_1N$ exceptional $(1/N)$-squares Q^* on any yz-parallel unit square that can be reached from a point of Q via a vector of length between 2^{r-1} and 2^r.

(iv) Irrespective of whether the two exceptional $(1/N)$-squares Q and Q^* are color-split or monochromatic, they determine a set of bad directions of surface area at most

$$c_3\left(\frac{1/N}{2^r}\right)^2$$

on the unit sphere.

Combining (i)–(iv), we conclude that the total surface area on the unit sphere of bad directions characterized by a vector of length between 2^{r-1} and 2^r is at most

$$9c_1N\cdot c_2 2^{3r}\cdot 9c_1N\cdot\frac{c_3}{2^{2r}N^2}=c_4 2^r.$$

Next, we use Lemma 3.3 with ε replaced by $\varepsilon/2$. Let $V\subset[-1,1]^2$ represent the at least $(1-\varepsilon/2)$-proportion of vectors $\mathbf{v}_0$ that do not lead to clustering. For any $\mathbf{v}_0\in V$ and any axis-parallel square $A\subset[0,1)^2$ of side length $1/N$, let

$$\Psi(\mathbf{v}_0;N;A;r)$$

denote the number of split-free chains in (3.11) of length between 2^{r-1} and 2^r. Then

$$\iint_V \Psi(\mathbf{v}_0; N; A; r)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \leqslant c_5 2^r. \tag{5.8}$$

Note that the vectors $\mathbf{v} = (1, \mathbf{v}_0) \in \mathbb{R}^3$, where $\mathbf{v}_0 \in [-1,1]^2$, represent directions $(x, y, z) \in \mathbb{R}^3$ with $x^2 + y^2 + z^2 = 1$ under the extra restriction $x \geqslant \max\{|y|, |z|\}$.

Given any $\sigma > 0$, write

$$\Theta(N; A; r; \sigma) = \{\mathbf{v}_0 \in V : \Psi(\mathbf{v}_0; N; A; r) \geqslant \sigma N\}. \tag{5.9}$$

Combining (5.8) and (5.9), we have

$$\lambda(\Theta(N; A; r; \sigma)) \leqslant \frac{1}{\sigma N} \iint_V \Psi(\mathbf{v}_0; N; A; r)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \leqslant \frac{c_5 2^r}{\sigma N}. \tag{5.10}$$

We next make the specification

$$\sigma = \sigma(r) = \frac{\gamma}{(m+1-r)^2} \tag{5.11}$$

for some parameter $\gamma > 0$. Then combining (5.10) and (5.11), we have

$$\lambda(\Theta(N; A; r; \sigma(r))) \leqslant \frac{c_5 (m+1-r)^2 2^r}{\gamma N} = \frac{4 c_5 (m+1-r)^2 2^{m-1}}{\gamma 2^{m+1-r} N}.$$

Combining this with (5.7), we deduce that

$$\sum_{r=1}^{m} \lambda(\Theta(N; A; r; \sigma(r))) \leqslant \frac{c_6 \eta}{\gamma} \sum_{r=1}^{m} \frac{(m+1-r)^2}{2^{m+1-r}} \leqslant \frac{c_7 \eta}{\gamma}.$$

It then follows that the set

$$\Theta(N; A; \gamma) = \bigcup_{r=1}^{m} \Theta(N; A; r; \sigma(r)) \tag{5.12}$$

satisfies

$$\lambda(\Theta(N; A; \gamma)) \leqslant \frac{c_7 \eta}{\gamma}. \tag{5.13}$$

Combining (5.9), (5.11) and (5.12), we see that

$$\Psi(\mathbf{v}_0; N; A, r) < \frac{\gamma N}{(m+1-r)^2}, \quad \mathbf{v}_0 \in V \setminus \Theta(N; A; \gamma), \quad r = 1, \ldots, m,$$

leading to an upper bound of the total number of split-free chains of length at most ηN in the form

$$\sum_{r=1}^{m} \Psi(\mathbf{v}_0; N; A, r) < \gamma N \sum_{r=1}^{m} \frac{1}{(m+1-r)^2} < 2\gamma N = \varepsilon N,$$

if we specify $\gamma = \varepsilon/2$. Moreover, by choosing $\eta = \eta(\varepsilon) = \varepsilon^2/c_7 = C_1 \varepsilon^2$, the upper bound in (5.13) becomes 2ε. Hence

$$\lambda(V \setminus \Theta(N; A; \gamma)) > 4\left(1 - \frac{\varepsilon}{2}\right) - 2\varepsilon = 4(1-\varepsilon).$$

This completes the proof. ☐

6. Theorems 2.1 and 2.2 and further comments

We begin by briefly describing a few points in the proofs of Theorems 2.1 and 2.2.

Sketch of Proof of Theorem 2.1. We essentially repeat the proof of Theorem 1.1, but with a modification necessitated by the switch from global to local repetition. This has the effect of weakening the conclusion from equidistribution for almost every direction to equidistribution only for a positive proportion of all directions. More precisely, we restrict the original proof from the whole torus to the repeated color-split neighborhood. Consider the middle-third of the critical line segment in the color-split neighborhood, shown in bold in Figures 2.2 and 2.3, that separates the red part from the green part. Again $T = T_{\mathbf{v}_0}$ denotes the discretization of the 1-direction geodesic in the n-cube 3-manifold with barriers, and we focus on the intersection points of this geodesic with the yz-parallel squares between neighboring cubes. Consider the set $\mathcal{V}_0 \subset [-1,1]^2$ of all vectors $\mathbf{v}_0$ that start from a point in the middle-third of the critical line segment and end at any point inside the color-split neighborhood. Clearly $\mathcal{V}_0$ has positive area, representing a positive proportion of all directions. Thus the first modification is to replace the set $[-1,1]^2$ of all directions by this substantially smaller set $\mathcal{V}_0$.

The second change concerns the definition of *substantial color-split* in Step 3 in Section 4, given by (4.3) and (4.4). Here we need to make the additional requirement that the small squares (4.3) intersect the middle-third of the critical line segment in the color-split neighborhood. Since this critical line segment has positive length, we can simply repeat the original argument of Step 3, and we end up losing only a constant factor. □

Sketch of Proof of Theorem 2.2. We essentially repeat the proof of Theorem 1.1, but in this case, it is even slightly simpler. Again $T = T_{\mathbf{v}_0}$ denotes the discretization of the 1-direction geodesic in the polycube 3-manifold with barriers, and we focus on the intersection points of this geodesic with the yz-parallel squares between neighboring cubes.

As in the proof of Theorem 1.1, the critical step is the double-step with color-split in the transition from $j^* - 1$ to $j^* + 1$. Again, we take advantage of the split-singularities. Here the existence of a private color-split neighborhood guarantees the simplest form of local split-increase, namely the trivial increase from 1 to 2. This leads to a local increase of the multiplicity function of S_0. □

We finish this paper by making a number of remarks.

The proof of ergodicity via contradiction is a *local* argument, by making use of arbitrarily small axis-parallel squares. A polygon behaves locally like a straight line segment, and the intersection of a small square with a straight line segment represents a trivial geometric problem. This explains why the barriers in the cases that we have discussed may be extended to much broader non-linear shapes, such as the shape of a circle or an ellipse, or of a smooth closed curve, or of a piecewise smooth closed curve. Such curves behave locally like their tangent lines, so our local proofs can be adapted to these situations.

Furthermore, the proof techniques generalize in any higher dimension, leading to analogous results.

Finally, we remark that Birkhoff's ergodic theorem, at the centre of our argument, is a pure existence result, and does not say anything about the speed of convergence in (3.5) and (3.6). For comparison, we note that the more specific Kronecker–Weyl equidistribution theorem has several time-quantitative extensions with explicit error terms; see, for instance, the Erdős–Turán–Koksma theorem in [2]. The error term characterizes perfectly the properties that prevent equidistribution. Unfortunately, we are not aware of any time-quantitative extension of Birkhoff's theorem.

In particular, it is an open problem to find any time-quantitative extensions of the results in this paper.

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Department of Mathematics, Rutgers University, Hill Center for the Mathematical Sciences, Piscataway NJ 08854, USA

E-mail address: `jbeck@math.rutgers.edu`

Department of Mathematics and Statistics, Macquarie University, Sydney NSW 2109, Australia

E-mail address: `william.chen@mq.edu.au`