SUPERDENSITY AND SUPER-MICRO-UNIFORMITY IN NON-INTEGRABLE FLAT SYSTEMS

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Abstract. We show that on any non-integrable finite polysquare translation surface, superdensity, an optimal form of time-quantitative density, leads to an optimal form of time-quantitative uniformity that we call super-micro-uniformity.

1. Introduction

Consider billiard in a polygon that satisfies the usual law of reflection in optics. It is trivial that uniformity of a half-infinite orbit always implies density, and that the converse is false. However, while density does not in general imply uniformity, we show here an interesting case when some form of time-quantitative density implies some form of time-quantitative uniformity.

The purpose of the present paper is to show how superdensity, an optimal form of time-quantitative density, implies an optimal form of time-quantitative uniformity that we call *super-micro-uniformity*. Here *super* refers to optimality and *micro* refers to microscopic scale.

To illustrate the latter, consider the irrational rotation sequence

$$\{k\alpha\}, \quad k = 1, 2, 3, \ldots, \tag{1.1}$$

of fractional parts of $k\alpha$ in the interval $[0, 1)$, where α is irrational. Let $I \subset [0, 1)$ be an arbitrary subinterval of length $1/2n$, and consider the first n elements of the sequence (1.1). Then the *expected number* of elements of this n-element set in I is clearly $1/2$, corresponding to n times the length of I. On the other hand, the *visiting number* $V_n(I)$ of I, the actual number of elements in I coming from this n-element set is clearly an integer, and so must differ from the expected number by at least $1/2$. We refer to this as the *trivial error*. Indeed, we have same phenomenon if the length of I is C/n, where $2C$ is an odd integer. Here the error is at least $1/2$, and the expected number C is in the constant range.

Given the first n elements of the infinite sequence (1.1), intervals of length C/n represent test sets in the microscopic scale. Here $C > 0$ is a fixed constant, and n may tend to infinity. The trivial error argument above implies that in the microscopic scale of C/n, we cannot expect *perfect local uniformity* in the sense that the ratio of the error term and the expected number tends to zero as C is fixed and n tends to infinity. To put it slightly differently, to have perfect local uniformity, it is necessary to have $C = C(n) \to \infty$ as $n \to \infty$.

It turns out that this necessary condition is sufficient to establish perfect local uniformity if α is badly approximable. This perfect local uniformity is what we call super-micro-uniformity. It has the intuitive meaning that the orbit exhibits uniformity already in the shortest possible subintervals. We have the following result on super-micro-uniformity of the irrational rotation sequence generated by a badly approximable α.

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Theorem A. *Let α be a badly approximable real number. For any subinterval $I \subset [0,1)$, let*

$$V_n(I) = \{k = 1, \ldots, n : \{k\alpha\} \in I\}$$

denote the visiting number of I with respect to the first n terms of the sequence (1.1)*. Then for every real number $\varepsilon > 0$, there exists a finite threshold $C_\varepsilon = C_\varepsilon(\alpha)$ such that for any subinterval I with length $|I| \geqslant C_\varepsilon/n$, the inequality*

$$||V_n(I) - n|I|| < \varepsilon n|I| \tag{1.2}$$

holds.

The proof of this result is a fairly routine exercise using continued fractions, so Theorem A is very possibly folklore. However, as we shall establish a more general result, we briefly outline the ideas here.

First of all, we recall that the convergents p_m/q_m of α give excellent rational approximation, in the sense that

$$\left|\alpha - \frac{p_m}{q_m}\right| < \frac{1}{q_m q_{m+1}},$$

so that

$$\left|k\alpha - \frac{kp_m}{q_m}\right| < \frac{k}{q_m q_{m+1}} \leqslant \frac{1}{q_{m+1}}, \quad k = 1, \ldots, q_m.$$

This implies that any segment of q_m consecutive terms of the sequence (1.1) is extremely uniformly distributed in the interval $[0,1)$.

To take advantage of this, it makes sense to look at the Ostrowski decomposition of integers, using the denominators of the convergents. For every integer N, we can write

$$N = \sum_{i=0}^{n} b_i q_i,$$

where n is the unique integer satisfying $q_n \leqslant N < q_{n+1}$, and the digits $b_0, b_1, \ldots, b_n$ satisfy

$$\begin{aligned} &b_0 \in \{0, 1, \ldots, a_1 - 1\}, \\ &b_i \in \{0, 1, \ldots, a_{i+1}\}, \qquad i = 1, \ldots, n, \\ &b_{i-1} = 0 \text{ if } b_i = a_{i+1}, \qquad i = 1, \ldots, n. \end{aligned}$$

where $a_1, \ldots, a_{n+1}$ are continued fraction digits of α.

Theorem A follows on combining these two ideas in a suitable way.

From the discrete super-micro-uniformity given by (1.2), it is easy to deduce that every half-infinite geodesic of badly approximable slope α is super-micro-uniform in the unit torus.

Note that geodesics on the unit torus is the simplest integrable system. If we consider geodesic flow on an arbitrary finite polysquare translation surface, then it is typically non-integrable.

Theorem 1. *Let $\mathcal{P}$ be a polysquare translation surface with b atomic squares, and let α be a badly approximable real number. Let $L_\alpha(t)$, $t \geqslant 0$, be a half-infinite geodesic with slope α, equipped with the usual arc-length parametrization. For any positive integer n, let $\mathcal{X}_n$ denote the set of the first n intersection points of $L_\alpha(t)$, $t \geqslant 0$, with the vertical edges of $\mathcal{P}$, and for any subinterval I of any vertical edge of $\mathcal{P}$, let*

$$V_n(I) = |I \cap \mathcal{X}_n|$$

denote the visiting number of I with respect to $\mathcal{X}_n$. Then for every real number $\varepsilon > 0$, there exists a finite threshold $C_\varepsilon = C_\varepsilon(\mathcal{P};\alpha)$ such that for any subinterval I

of any vertical edge of $\mathcal{P}$ with length $|I| \geqslant C_\varepsilon/n$, the inequality

$$\left| V_n(I) - \frac{n|I|}{b} \right| < \varepsilon \frac{n|I|}{b}$$

holds. In other words, we have super-micro-uniformity.

The remainder of the paper is devoted to proving this result.

We require a superdensity result in our earlier papers [1, 2]. Let $\mathcal{P}$ be a polysquare translation surface with b atomic squares, and let α be a badly approximable real number. Then there exists a finite superdensity threshold $c_0 = c_0(\mathcal{P};\alpha)$ such that for every integer $m \geqslant 1$, any geodesic segment of slope α and length $c_0 m$ gets $(1/m)$-close to every point of $\mathcal{P}$.

2. Iteration process: step 0

Let C be a constant satisfying $1 < C < n$. Let $\mathcal{I}_n(\mathcal{P};C)$ denote the collection of any subinterval I of any vertical edge of $\mathcal{P}$ with length $|I| = C/n$, and let $I_0, I_1 \in \mathcal{I}_n(\mathcal{P};C)$ be subintervals satisfying

$$V_n(I_0) = \min_{I \in \mathcal{I}_n(\mathcal{P};C)} |I \cap \mathcal{X}_n| \quad \text{and} \quad V_n(I_1) = \max_{I \in \mathcal{I}_n(\mathcal{P};C)} |I \cap \mathcal{X}_n|,$$

so that I_0 and I_1 have respectively the smallest and largest visiting numbers with respect to $\mathcal{X}_n$ among all the subintervals I under consideration. It is clear that

$$|I_0 \cap \mathcal{X}_n| \leqslant \frac{C}{b} \leqslant |I_1 \cap \mathcal{X}_n|. \tag{2.1}$$

Let the real number ε satisfy $0 < \varepsilon < 1/2$. We have two cases:

Case A. We have

$$\frac{|I_0 \cap \mathcal{X}_n|}{|I_1 \cap \mathcal{X}_n|} \geqslant 1 - \varepsilon. \tag{2.2}$$

Case B. We have

$$\frac{|I_0 \cap \mathcal{X}_n|}{|I_1 \cap \mathcal{X}_n|} < 1 - \varepsilon. \tag{2.3}$$

We shall postpone the analysis of Case A to Section 6.

To complete the proof of Theorem 1, we shall show that Case B, where (2.3) holds, is not possible. Indeed, we shall show that (2.3) leads to a contradiction. The proof is rather long, and involves a complicated iteration process, with two possibilities at each step. We shall derive the necessary contradiction by showing that at some stage of this process, neither possibility is valid.

We divide the interval I_0 into 3 subintervals of equal length, and denote by $I_0(0)$ one of them with the minimum intersection with the set $\mathcal{X}_n$, so that

$$\frac{|I_0(0) \cap \mathcal{X}_n|}{|I_0(0)|} \leqslant \frac{|I_0 \cap \mathcal{X}_n|}{|I_0|}. \tag{2.4}$$

The α-flow moves $I_0(0)$ forward. Consider the first time that the image hits a vertex and splits into 2 parts, and denote by $I_0(1)$ the larger part. Clearly $I_0(1)$ has length

$$|I_0(1)| \geqslant \frac{|I_0(0)|}{2} \geqslant \frac{C}{6n},$$

and one of its endpoints is a vertex of $\mathcal{P}$. The α-flow moves $I_0(1)$ forward. Consider the first time that the image hits a vertex and splits into 2 parts, and denote by $I_0(2)$ the larger part. Clearly $I_0(2)$ has length

$$|I_0(2)| \geqslant \frac{|I_0(1)|}{2} \geqslant \frac{C}{12n},$$

and one of its endpoints is a vertex of $\mathcal{P}$. The α-flow moves $I_0(2)$ forward, and so on. We obtain a sequence $I_0(j)$, $j=0,1,2,3,\ldots$, of intervals such that the length

$$|I_0(j)| \geqslant \frac{C}{2^j 3n}, \quad j=0,1,2,3,\ldots,$$

and each interval $I_0(j)$, $j=1,2,3,\ldots$, has an endpoint which is a vertex of $\mathcal{P}$. We next apply the reverse α-flow to send each of these intervals $I_0(j)$, $j=1,2,3,\ldots$, back to a subinterval $I_0^*(j)$ of I_0, giving rise to a nested sequence

$$I_0 \supset I_0(0) \supset I_0^*(1) \supset I_0^*(2) \supset I_0^*(3) \supset \ldots.$$

Furthermore, the α-flow moves each subinterval $I_0^*(j)$, $j=1,2,3,\ldots$, of I_0 to $I_0(j)$ split-free.

Let j_0 denote the smallest integer $j=1,2,3,\ldots$ such that $\mathrm{length}(j_0)$, the length of the α-flow moving $I_0^*(j)$ to $I_0(j)$, satisfies

$$\mathrm{length}(j_0) > \frac{c_0}{|I_1|/3} = \frac{3c_0 n}{C}, \tag{2.5}$$

where $c_0 = c_0(\mathcal{P};\alpha)$ is the superdensity threshold. Superdensity and (2.5) then imply that the α-flow moving $I_0^*(j_0)$ forward must intersect the middle-third of the interval I_1 before reaching $I_0(j_0)$, and let $I_1^*(j_0)$ denote the image of $I_0^*(j_0)$ under the α-flow on the vertical edge of $\mathcal{P}$ that contains I_1, as shown in Figure 1. Clearly we have

$$I_1^*(j_0) \subset I_1 \quad \text{and} \quad \frac{C}{3n} \geqslant |I_1^*(j_0)| = |I_0(j_0)| \geqslant \frac{C}{2^{j_0} 3n}. \tag{2.6}$$

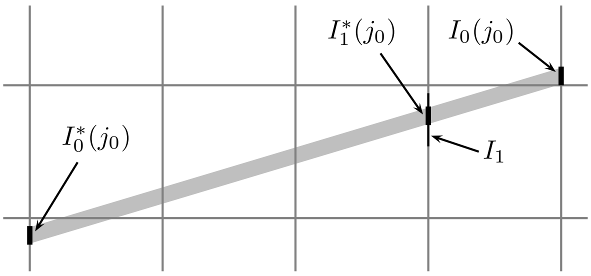

Figure 1: the subinterval $I_1^*(j_0)$ on the same vertical edge of $\mathcal{P}$ as I_1

It also follows from the definition of j_0 that $\mathrm{length}(j_0-1)$, the length of the α-flow moving $I_0^*(j-1)$ to $I_0(j-1)$, satisfies

$$\mathrm{length}(j_0-1) \leqslant \frac{c_0}{|I_1|/3} = \frac{3c_0 n}{C}. \tag{2.7}$$

Consider the sequence $q_1, q_2, q_3, \ldots$ of the denominators of the convergents of α. There exists a unique integer h such that

$$q_h \leqslant \frac{1}{|I_0(j_0-1)|} < q_{h+1} < (A+1)q_h, \tag{2.8}$$

where A is an upper bound of the continued fraction digits of α.

Lemma 1. *Suppose that q_h is the denominator of a convergent of α and I is an interval of real numbers. Consider the equation*

$$\{\beta + k\alpha\} = 0, \quad \beta \in I, \quad k = 1,\ldots,q_h.$$

(i) *If I has length $|I| \leqslant 1/q_h$, then the equation has at most 3 solutions (β,k).*
(ii) *If I has length $|I| \geqslant 3/q_h$, then the equation has at least 1 solution (β,k).*

Proof. It is well known that the convergent p_h/q_h of α satisfies

$$\left|\alpha-\frac{p_h}{q_h}\right|<\frac{1}{q_hq_{h+1}}.$$

It follows that

$$\left|k\alpha-\frac{kp_h}{q_h}\right|<\frac{k}{q_hq_{h+1}}\leqslant\frac{1}{q_{h+1}},\quad k=1,\ldots,q_h. \tag{2.9}$$

Consider the mapping

$$k\alpha\mapsto\left\{\frac{kp_h}{q_h}\right\}=\frac{\ell(k)}{q_h},\quad k=1,\ldots,q_h. \tag{2.10}$$

Since p_h and q_h are coprime, the numbers kp_h, $k=1,\ldots,q_h$, form a complete set of residues modulo q_h, so that

$$\{\ell(k):k=1,\ldots,q_h\}=\{0,1,\ldots,q_h-1\}. \tag{2.11}$$

Combining (2.9)–(2.11), we conclude that there is a 1-1 correspondence between the collection of values $k\alpha$, $k=1,\ldots,q_h$, and the collection of intervals

$$\mathfrak{I}_0=\left[0,\frac{1}{q_{h+1}}\right)\cup\left(1-\frac{1}{q_{h+1}},1\right), \tag{2.12}$$

$$\mathfrak{I}_\ell=\left(\frac{\ell}{q_h}-\frac{1}{q_{h+1}},\frac{\ell}{q_h}+\frac{1}{q_{h+1}}\right),\quad \ell=1,\ldots,q_h-1. \tag{2.13}$$

In particular, we have

$$k\alpha\in\mathfrak{I}_{\ell(k)}\bmod 1,\quad k=1,\ldots,q_h. \tag{2.14}$$

(i) Suppose that $|I|\leqslant 1/q_h$ and β_0 is the lower endpoint of the interval I. It follows from (2.14) that for every $\beta\in I$, we have

$$\{\beta+k\alpha\}\in\mathfrak{I}^*_{\ell(k)}(\beta_0)\bmod 1,\quad k=1,\ldots,q_h,$$

where the intervals $\mathfrak{I}^*_\ell(\beta_0)\subset[0,1)$, $\ell=0,1,\ldots,q_h-1$, satisfy

$$\mathfrak{I}^*_0(\beta_0)=\left[\beta_0,\frac{1}{q_{h+1}}+\beta_0+\frac{1}{q_h}\right)\cup\left(1-\frac{1}{q_{h+1}}+\beta_0,1+\beta_0\right)\bmod 1, \tag{2.15}$$

$$\mathfrak{I}^*_\ell(\beta_0)=\left(\frac{\ell}{q_h}-\frac{1}{q_{h+1}}+\beta_0,\frac{\ell}{q_h}+\frac{1}{q_{h+1}}+\beta_0+\frac{1}{q_h}\right)\bmod 1,\quad \ell=1,\ldots,q_h-1. \tag{2.16}$$

Note now that each of the q_h intervals in (2.15) and (2.16) is of length less than $3/q_h$, so can contain at most 1 integer. Together they form a q_h-cycle through translation by $1/q_h$, so that at most 3 of them can contain an integer.

(ii) Suppose that $|I|\geqslant 3/q_h$ and β_0 is the lower endpoint of the interval I. It follows from (2.14) that

$$\{\beta_0+k\alpha\}\in\mathfrak{I}_{\ell(k)}+\beta_0,\quad k=1,\ldots,q_h,$$

where the intervals $\mathfrak{I}_\ell+\beta_0$, $\ell=0,1,\ldots,q_h-1$, are the translates of the intervals (2.12) and (2.13) modulo 1. Consider the collection of intervals

$$\mathfrak{I}^{**}_\ell(\beta_0)=\left[\beta_0+\frac{\ell+1}{q_h},\beta_0+\frac{\ell+2}{q_h}\right]\bmod 1,\quad \ell=0,\ldots,q_h-1. \tag{2.17}$$

It is then not difficult to see that

$$\{\{\beta+k\alpha\}:\beta\in I\}\supset\mathfrak{I}^{**}_{\ell(k)}(\beta_0),\quad k=1,\ldots,q_h. \tag{2.18}$$

Note finally that the union of the q_h sets in (2.17) is the whole interval $[0,1)$. It follows from this and (2.18) that $\{\beta+k\alpha\}=0$ for some $\beta\in I$ and $k=1,\ldots,q_h$. □

Let us now focus on the transportation process as the α-flow moves the interval $I_0^*(j_0-1)$ to the interval $I_0(j_0-1)$. It follows from the first inequality of (2.8) and Lemma 1(i) that, among any q_h consecutive intersections with the vertical edges of $\mathcal{P}$, the flow hits a vertex at most 3 times. Now the common length of a geodesic segment of slope α between consecutive intersections with the vertical edges of $\mathcal{P}$ is $\sqrt{1+\alpha^2}$. In view of the bound (2.7), we clearly have

$$\frac{(j_0-1)q_h\sqrt{1+\alpha^2}}{3} \leqslant \text{length}(j_0-1) \leqslant \frac{3c_0n}{C}. \tag{2.19}$$

Combining this with the trivial upper bound for $|I_0(j_0-1)| \leqslant C/n$ and the second inequality in (2.8), we obtain

$$j_0-1 \leqslant \frac{9c_0n}{Cq_h\sqrt{1+\alpha^2}} \leqslant \frac{9c_0n(A+1)|I_0(j_0-1)|}{C\sqrt{1+\alpha^2}} \leqslant \frac{9c_0(A+1)}{\sqrt{1+\alpha^2}} = c_1, \tag{2.20}$$

where the constant c_1 is independent of the parameters n, C and ε. Combining (2.6) and (2.20), we deduce that

$$\frac{1}{3}|I_1| \geqslant |I_1^*(j_0)| \geqslant \frac{c_2C}{n} = c_2|I_1|, \tag{2.21}$$

where the constant

$$c_2 = \frac{1}{2^{c_1+1}3} \tag{2.22}$$

is independent of the parameters n, C and ε.

Consider again the sequence $q_1, q_2, q_3, \ldots$ of the denominators of the convergents of α. There exists a unique integer h^* such that

$$q_{h^*} \geqslant \frac{3}{|I_0(j_0)|} > q_{h^*-1}. \tag{2.23}$$

Combining the continued fraction inequality $q_{h^*} < (A+1)q_{h^*-1}$ with the inequality $|I_0(j_0)| \geqslant |I_0(j_0-1)|/2$ and (2.8), we have

$$q_{h^*} \leqslant \frac{3(A+1)}{|I_0(j_0)|} \leqslant \frac{6(A+1)}{|I_0(j_0-1)|} \leqslant 6(A+1)^2q_h. \tag{2.24}$$

Let us now focus on the transportation process as the α-flow moves the interval $I_0^*(j_0)$ to the interval $I_0(j_0)$ via the interval $I_1^*(j_0)$. It follows from the first inequality in (2.23) and Lemma 1(ii) that, among any q^{h^*} consecutive intersections with the vertical edges of $\mathcal{P}$, the flow hits a vertex at least 1 time. Recall that the common length of a geodesic segment of slope α between consecutive intersections with the vertical edges of $\mathcal{P}$ is $\sqrt{1+\alpha^2}$. Thus

$$j_0q_{h^*}\sqrt{1+\alpha^2} \geqslant \text{length}(j_0). \tag{2.25}$$

Combining (2.19), (2.24) and (2.25), we conclude that for $j_0 \geqslant 2$, we have

$$\text{length}(j_0) \leqslant 36(A+1)^2\,\text{length}(j_0-1) \leqslant \frac{108(A+1)^2c_0n}{C}. \tag{2.26}$$

For $j_0 = 1$, combining (2.8), (2.24) and (2.25), we have

$$\text{length}(j_0) \leqslant 6(A+1)^2q_h\sqrt{1+\alpha^2} \leqslant \frac{6(A+1)^2\sqrt{1+\alpha^2}}{|I_0(0)|} = \frac{6(A+1)^2n\sqrt{1+\alpha^2}}{C}.$$

We may assume without loss of generality that $\alpha > 1$. Then

$$\sqrt{1+\alpha^2} < 2\alpha < 2(a_0+1) \leqslant 2(A+1),$$

where a_0 is the first continued fraction digit of α. It follows that for $j_0 = 1$, we have

$$\text{length}(j_0) \leqslant \frac{12(A+1)^3n}{C}. \tag{2.27}$$

We may assume that the superdensity threshold $c_0 \geqslant 1$. Then it follows from (2.26) and (2.27) that the inequality

$$\text{length}(j_0) \leqslant \frac{108(A+1)^3 c_0 n}{C} \tag{2.28}$$

always holds.

We have two cases:

Case 1. We have

$$\frac{|I_1^*(j_0) \cap \mathcal{X}_n|}{|I_1^*(j_0)|} \geqslant \left(1 - \frac{\varepsilon}{2}\right) \frac{|I_1 \cap \mathcal{X}_n|}{|I_1|}. \tag{2.29}$$

Case 2. We have

$$\frac{|I_1^*(j_0) \cap \mathcal{X}_n|}{|I_1^*(j_0)|} < \left(1 - \frac{\varepsilon}{2}\right) \frac{|I_1 \cap \mathcal{X}_n|}{|I_1|}. \tag{2.30}$$

Before we study these cases in detail, we first give some heuristics to explain the underlying ideas. If Case 1 holds, then we show that the subinterval $I_0^*(j_0) \subset I_0$ exhibits a surplus density of points of $\mathcal{X}_n$ compared to I_0. Removing this subinterval, the remaining part of I_0 then exhibits a deficit density of points of $\mathcal{X}_n$ compared to I_0. If Case 2 holds, then the subinterval $I_1^*(j_0) \subset I_1$ exhibits a deficit density of points of $\mathcal{X}_n$ compared to I_1. Removing this subinterval, the remaining part of I_1 then exhibits a surplus density of points of $\mathcal{X}_n$ compared to I_1. Thus we either obtain a subinterval of I_0 that exhibits deficit density of points of $\mathcal{X}_n$ compared to I_0, or obtain a subinterval of I_1 that exhibits surplus density of points of $\mathcal{X}_n$ compared to I_1. We then repeat the analysis on intervals of length equal to the length of this subinterval, and this sets up an iteration process. We then show that this iteration has to terminate after a finite number of steps, and this gives the necessary contradiction.

Let $I^\star$ be an arbitrary subinterval of a vertical edge of $\mathcal{P}$. Since geodesic flow on $\mathcal{P}$ modulo one is geodesic flow on the unit torus, and the slope α is badly approximable with digit upper bound A, we have

$$|I^\star \cap \mathcal{X}_n| \leqslant An|I^\star| + 1. \tag{2.31}$$

2.1. **Case 1: density decrease.** Suppose that the inequality (2.29) holds. To find a lower bound to $|I_0^*(j_0) \cap \mathcal{X}_n|$, we consider the transportation process as the α-flow moves the interval $I_0^*(j_0)$ to the interval $I_1^*(j_0)$. The length of this flow is at most length(j_0), so the flow intersects the vertical edges of $\mathcal{P}$ at most

$$\frac{\text{length}(j_0)}{\sqrt{1+\alpha^2}} + 1 < 2\,\text{length}(j_0)$$

times, but these intersection points are not necessarily points of $\mathcal{X}_n$. An analog of (2.31) then says that the number of these intersection points that are the images of points in $I_0^*(j_0)$ under the α-flow is at most

$$2A\,\text{length}(j_0)|I_0^*(j_0)| + 1.$$

It follows that

$$|I_0^*(j_0) \cap \mathcal{X}_n| \geqslant |I_1^*(j_0) \cap \mathcal{X}_n| - 2A\,\text{length}(j_0)|I_0^*(j_0)| - 1. \tag{2.32}$$

Note that (2.29) gives

$$|I_1^*(j_0) \cap \mathcal{X}_n| \geqslant \left(1 - \frac{\varepsilon}{2}\right) |I_1^*(j_0)| \frac{|I_1 \cap \mathcal{X}_n|}{|I_1|}, \tag{2.33}$$

while (2.28) and $|I_1| = C/n$ give

$$2A\,\mathrm{length}(j_0) \leqslant \frac{216A(A+1)^3c_0}{|I_1|}. \tag{2.34}$$

Combining (2.32)–(2.34) and noting that $|I_1 \cap \mathcal{X}_n| \geqslant C/b$ and $|I_0^*(j_0)| = |I_1^*(j_0)|$, we obtain the inequality

$$|I_0^*(j_0) \cap \mathcal{X}_n| \geqslant \left(1 - \frac{3\varepsilon}{4}\right) |I_0^*(j_0)| \frac{|I_1 \cap \mathcal{X}_n|}{|I_1|}, \tag{2.35}$$

provided that each of the last two terms on the right in (2.32) is bounded by

$$\frac{\varepsilon}{8}|I_0^*(j_0)|\frac{|I_1 \cap \mathcal{X}_n|}{|I_1|}.$$

This is guaranteed if we ensure that

$$C \geqslant \frac{1728A(A+1)^3c_0b}{\varepsilon} \quad \text{and} \quad C \geqslant \frac{8b}{c_2\varepsilon}, \tag{2.36}$$

the latter in view of (2.21). Combining (2.3) and (2.35) now leads to the inequality

$$\begin{aligned} |I_0^*(j_0) \cap \mathcal{X}_n| &\geqslant \left(1 - \frac{3\varepsilon}{4}\right) (1-\varepsilon)^{-1} |I_0^*(j_0)| \frac{|I_0 \cap \mathcal{X}_n|}{|I_0|} \\ &> \left(1 + \frac{\varepsilon}{4}\right) |I_0^*(j_0)| \frac{|I_0 \cap \mathcal{X}_n|}{|I_0|}, \end{aligned} \tag{2.37}$$

provided that (2.36) holds.

There are at most two subintervals $I_{0,1}, I_{0,2} \subset I_0$ such that

$$I_0 = I_0^*(j_0) \cup I_{0,1} \cup I_{0,2}. \tag{2.38}$$

Removing the interval $I_0^*(j_0)$ and combining (2.37) and (2.38), we obtain

$$\begin{aligned} &|I_{0,1} \cap \mathcal{X}_n| + |I_{0,2} \cap \mathcal{X}_n| = |I_0 \cap \mathcal{X}_n| - |I_0^*(j_0) \cap \mathcal{X}_n| \\ &\quad < |I_0 \cap \mathcal{X}_n| \left(1 - \left(1 + \frac{\varepsilon}{4}\right) \frac{|I_0^*(j_0)|}{|I_0|}\right) \leqslant |I_0 \cap \mathcal{X}_n| \left(\frac{|I_{0,1}| + |I_{0,2}|}{|I_0|} - \frac{c_2\varepsilon}{4}\right), \end{aligned} \tag{2.39}$$

noting that (2.21) implies $|I_0^*(j_0)|/|I_0| \geqslant c_2$. There are two possibilities, either

$$\frac{\min\{|I_{0,1}|, |I_{0,2}|\}}{|I_0|} < \frac{c_2\varepsilon}{8}, \tag{2.40}$$

or

$$\frac{\min\{|I_{0,1}|, |I_{0,2}|\}}{|I_0|} \geqslant \frac{c_2\varepsilon}{8}. \tag{2.41}$$

If (2.40) holds, then we may assume without loss of generality that

$$|I_{0,1}| \geqslant |I_{0,2}|, \quad \text{so that} \quad \frac{|I_{0,2}|}{|I_0|} < \frac{c_2\varepsilon}{8},$$

and so it follows from (2.39) that

$$|I_{0,1} \cap \mathcal{X}_n| \leqslant |I_0 \cap \mathcal{X}_n| \left(\frac{|I_{0,1}|}{|I_0|} - \frac{c_2\varepsilon}{8}\right). \tag{2.42}$$

On the other hand, if (2.41) holds, then since it follows from (2.39) that

$$|I_{0,1} \cap \mathcal{X}_n| + |I_{0,2} \cap \mathcal{X}_n| \leqslant |I_0 \cap \mathcal{X}_n| \left(\frac{|I_{0,1}|}{|I_0|} - \frac{c_2\varepsilon}{8}\right) + |I_0 \cap \mathcal{X}_n| \left(\frac{|I_{0,2}|}{|I_0|} - \frac{c_2\varepsilon}{8}\right),$$

we may assume without loss of generality that (2.42) holds again. Note now that (2.42) leads to the inequality

$$|I_{0,1}| \geqslant \frac{c_2\varepsilon}{8}|I_0|,$$

as well as the inequality

$$\frac{|I_{0,1}\cap\mathcal{X}_n|}{|I_{0,1}|}\leqslant\frac{|I_0\cap\mathcal{X}_n|}{|I_0|}\left(1-\frac{c_2\varepsilon}{8}\frac{|I_0|}{|I_{0,1}|}\right)\leqslant\left(1-\frac{c_2\varepsilon}{8}\right)\frac{|I_0\cap\mathcal{X}_n|}{|I_0|}. \tag{2.43}$$

Thus the switch from I_0 to $I_{0,1}$ leads to density decrease by a factor of $1-c_2\varepsilon/8$.

The ratio $|I_0|/|I_{0,1}|$ is not necessarily an integer. To overcome this issue, we shall replace $I_{0,1}$ by a suitable subinterval at the expense of part of the density decrease.

We shall use the following almost trivial observation a number of times.

Lemma 2. *Suppose that I is a finite interval of real numbers, $\mathcal{Y}\subset I$ is a finite subset with m elements, and z is a real number satisfying $0<z\leqslant|I|$.*

(i) *Then there exists a subinterval $I'\subset I$ of length $|I'|=z$ such that*

$$|I'\cap\mathcal{Y}|\leqslant m\frac{z}{|I|-z}.$$

(ii) *Suppose further that there exists an integer $B\geqslant 1$ such that every subinterval $I^\dagger\subset I$ satisfies*

$$|I^\dagger\cap\mathcal{Y}|\leqslant Bm\frac{|I^\dagger|}{|I|}+1. \tag{2.44}$$

Then there exists a subinterval $I''\subset I$ of length $|I''|=z$ such that

$$|I''\cap\mathcal{Y}|\geqslant(m-1)\frac{z}{|I|}-Bm\left(\frac{z}{|I|}\right)^2.$$

Proof. Write $|I|=kz+w$, where $k\geqslant 1$ is an integer and $0\leqslant w<z$. We partition the interval I into a union

$$I=J_0\cup J_1\cup\ldots\cup J_k,\quad |J_0|=w,\quad |J_1|=\ldots=|J_k|=z.$$

(i) Among the intervals $J=J_1,\ldots,J_k$, let I' be one for which $|J\cap\mathcal{Y}|$ is minimal. Observing the inequality

$$k=\frac{|I|-w}{z}>\frac{|I|-z}{z},$$

we deduce that

$$|I'\cap\mathcal{Y}|\leqslant\frac{|\mathcal{Y}|}{k}\leqslant m\frac{z}{|I|-z}.$$

(ii) Among the intervals $J=J_1,\ldots,J_k$, let I'' be one for which $|J\cap\mathcal{Y}|$ is maximal. Applying (2.44) to the subinterval I_0, we have

$$|I_0\cap\mathcal{Y}|<\frac{Bmz}{|I|}+1.$$

Observing this and the inequality

$$k=\frac{|I|-w}{z}\leqslant\frac{|I|}{z},$$

we deduce that

$$|I''\cap\mathcal{Y}|\geqslant\frac{|\mathcal{Y}|-|I_0\cap\mathcal{Y}|}{k}\geqslant\frac{z}{|I|}\left(m-\frac{Bmz}{|I|}-1\right)=(m-1)\frac{z}{|I|}-Bm\left(\frac{z}{|I|}\right)^2.$$

This completes the proof. □

Let h_0 be the unique integer satisfying

$$h_0-1<\frac{16|I_0|}{c_2\varepsilon|I_{0,1}|}\leqslant h_0, \tag{2.45}$$

so that

$$\frac{|I_0|}{h_0}\leqslant\frac{c_2\varepsilon}{16}|I_{0,1}|\leqslant|I_{0,1}|, \tag{2.46}$$

provided that ε is sufficiently small.

We now apply Lemma 2(i) with $I = I_{0,1}$, $\mathcal{Y} = I_{0,1} \cap \mathcal{X}_n$ and $z = |I_0|/h_0$. Then there exists a subinterval $I_0(\star) \subset I_{0,1}$ with $|I_0(\star)| = z$ such that

$$|I_0(\star) \cap \mathcal{X}_n| \leqslant |I_{0,1} \cap \mathcal{X}_n| \frac{z}{|I_{0,1}| - z}.$$

Combining this with the estimate (2.43), we deduce that

$$\frac{|I_0(\star) \cap \mathcal{X}_n|}{|I_0(\star)|} \leqslant \frac{|I_{0,1} \cap \mathcal{X}_n|}{|I_{0,1}|} \frac{|I_{0,1}|}{|I_{0,1}| - z} \leqslant \left(1 - \frac{c_2\varepsilon}{8}\right) \frac{|I_0 \cap \mathcal{X}_n|}{|I_0|} \frac{|I_{0,1}|}{|I_{0,1}| - z}. \tag{2.47}$$

Note from (2.46) that

$$\frac{|I_{0,1}|}{|I_{0,1}| - z} \leqslant \left(1 - \frac{c_2\varepsilon}{16}\right)^{-1}.$$

Combining this with (2.47), we deduce that

$$\frac{|I_0(\star) \cap \mathcal{X}_n|}{|I_0(\star)|} \leqslant \left(1 - \frac{c_2\varepsilon}{8}\right) \left(1 - \frac{c_2\varepsilon}{16}\right)^{-1} \frac{|I_0 \cap \mathcal{X}_n|}{|I_0|} \leqslant \left(1 - \frac{c_2\varepsilon}{16}\right) \frac{|I_0 \cap \mathcal{X}_n|}{|I_0|}. \tag{2.48}$$

Thus the switch from I_0 to $I_0(\star)$ leads to density decrease by a factor of $1 - c_2\varepsilon/16$, with the added benefit that the ratio $|I_0|/|I_0(\star)|$ is an integer h_0.

To obtain a subinterval of I_1 of the same length as $I_0(\star)$, we next divide I_1 into h_0 equal parts, and denote by $I_1(\star)$ one of these subintervals with the maximum intersection with the set $\mathcal{X}_n$. Then

$$\frac{|I_1(\star) \cap \mathcal{X}_n|}{|I_1(\star)|} \geqslant \frac{|I_1 \cap \mathcal{X}_n|}{|I_1|}. \tag{2.49}$$

Note that

$$|I_1(\star)| = |I_0(\star)| = \frac{C_1}{n}, \quad C_1 = \frac{C}{h_0}, \quad h_0 < c_3(\varepsilon), \tag{2.50}$$

where the constant $c_3(\varepsilon) > 0$ is independent of n and C.

2.2. **Case 2: density increase.** Suppose that the inequality (2.30) holds. There are at most two subintervals $I_{1,1}, I_{1,2} \subset I_1$ such that

$$I_1 = I_1^*(j_0) \cup I_{1,1} \cup I_{1,2}. \tag{2.51}$$

Removing the interval $I_1^*(j_0)$ and combining (2.30) and (2.51), we obtain

$$\begin{aligned} |I_{1,1} \cap \mathcal{X}_n| + |I_{1,2} \cap \mathcal{X}_n| &= |I_1 \cap \mathcal{X}_n| - |I_1^*(j_0) \cap \mathcal{X}_n| \\ &> |I_1 \cap \mathcal{X}_n| \left(1 - \left(1 - \frac{\varepsilon}{2}\right) \frac{|I_1^*(j_0)|}{|I_1|}\right) \geqslant |I_1 \cap \mathcal{X}_n| \left(\frac{|I_{1,1}| + |I_{1,2}|}{|I_1|} + \frac{c_2\varepsilon}{2}\right), \end{aligned} \tag{2.52}$$

noting that (2.21) implies $|I_1^*(j_0)|/|I_1| \geqslant c_2$. There are two possibilities, either

$$\frac{\min\{|I_{1,1}|, |I_{1,2}|\}}{|I_1|} < \frac{c_2\varepsilon}{5Ab}, \tag{2.53}$$

or

$$\frac{\min\{|I_{1,1}|, |I_{1,2}|\}}{|I_1|} \geqslant \frac{c_2\varepsilon}{5Ab}. \tag{2.54}$$

If (2.53) holds, then we may assume without loss of generality that

$$|I_{1,1}| \geqslant |I_{1,2}|, \quad \text{so that} \quad \frac{|I_{1,1}|}{|I_1|} \geqslant \frac{1}{3} \quad \text{and} \quad \frac{|I_{1,2}|}{|I_1|} < \frac{c_2\varepsilon}{5Ab}. \tag{2.55}$$

Combining (2.1), (2.31) and (2.55), we obtain

$$|I_{1,2} \cap \mathcal{X}_n| \leqslant An|I_{1,2}| + 1 \leqslant \frac{c_2\varepsilon}{5b} n|I_1| + 1 \leqslant \frac{c_2\varepsilon}{4b} n|I_1| \leqslant \frac{c_2\varepsilon}{4} |I_1 \cap \mathcal{X}_n|,$$

provided that n is sufficiently large. Substituting this into (2.52), we deduce that

$$|I_{1,1} \cap \mathcal{X}_n| \geqslant |I_1 \cap \mathcal{X}_n| \left(\frac{|I_{1,1}|}{|I_1|} + \frac{c_2 \varepsilon}{4} \right). \tag{2.56}$$

On the other hand, if (2.54) holds, then since it follows from (2.52) that

$$|I_{1,1} \cap \mathcal{X}_n| + |I_{1,2} \cap \mathcal{X}_n| \geqslant |I_1 \cap \mathcal{X}_n| \left(\frac{|I_{1,1}|}{|I_1|} + \frac{c_2 \varepsilon}{4} \right) + |I_1 \cap \mathcal{X}_n| \left(\frac{|I_{1,2}|}{|I_1|} + \frac{c_2 \varepsilon}{4} \right),$$

we may assume without loss of generality that (2.56) holds again. Note now that

$$|I_{1,1}| \geqslant \frac{c_2 \varepsilon}{5Ab} |I_1|, \tag{2.57}$$

provided that ϵ is sufficiently small, and (2.56) leads to the inequality

$$\frac{|I_{1,1} \cap \mathcal{X}_n|}{|I_{1,1}|} \geqslant \frac{|I_1 \cap \mathcal{X}_n|}{|I_1|} \left(1 + \frac{c_2 \varepsilon}{4} \frac{|I_1|}{|I_{1,1}|} \right) \geqslant \left(1 + \frac{c_2 \varepsilon}{4} \right) \frac{|I_1 \cap \mathcal{X}_n|}{|I_1|}. \tag{2.58}$$

Thus the switch from I_1 to $I_{1,1}$ leads to density increase by a factor $1 + c_2\varepsilon/4$.

The ratio $|I_1|/|I_{1,1}|$ is not necessarily an integer. To overcome this issue, we shall replace $I_{1,1}$ by a suitable subinterval at the expense of part of the density increase.

Let h_0 be the unique integer satisfying

$$h_0 - 1 < \frac{24Ab|I_1|}{c_2 \varepsilon |I_{1,1}|} \leqslant h_0, \tag{2.59}$$

so that

$$\frac{|I_1|}{h_0} \leqslant \frac{c_2 \varepsilon}{24Ab} |I_{1,1}| \leqslant |I_{1,1}|, \tag{2.60}$$

provided that ϵ is sufficiently small.

We now apply Lemma 2(ii) with $I = I_{1,1}$, $\mathcal{Y} = I_{1,1} \cap \mathcal{X}_n$ and $z = |I_1|/h_0$. Note that in view of (2.31), for every subinterval $I^\dagger \subset I_{1,1}$, we have

$$|I^\dagger \cap \mathcal{Y}| = |I^\dagger \cap \mathcal{X}_n| \leqslant An|I^\dagger| + 1. \tag{2.61}$$

On the other hand, it follows from (2.1), (2.58) and $|I_1| = C/n$ that

$$\frac{|I_{1,1} \cap \mathcal{X}_n|}{|I_{1,1}|} \geqslant \frac{|I_1 \cap \mathcal{X}_n|}{|I_1|} \geqslant \frac{n}{b}. \tag{2.62}$$

Combining (2.61) and (2.62), we have

$$|I^\dagger \cap \mathcal{Y}| \leqslant Ab|I_{1,1} \cap \mathcal{X}_n| \frac{|I^\dagger|}{|I_{1,1}|} + 1,$$

so that Lemma 2(ii) is valid with the constant $B = Ab$. It follows that there exists a subinterval $I_1(\star) \subset I_{1,1}$ with $|I_1(\star)| = z$ such that

$$|I_1(\star) \cap \mathcal{X}_n| \geqslant (|I_{1,1} \cap \mathcal{X}_n| - 1) \frac{z}{|I_{1,1}|} - Ab|I_{1,1} \cap \mathcal{X}_n| \left(\frac{z}{|I_{1,1}|} \right)^2.$$

Combining this with (2.60), we have

$$\begin{aligned} \frac{|I_1(\star) \cap \mathcal{X}_n|}{|I_1(\star)|} &\geqslant \frac{|I_{1,1} \cap \mathcal{X}_n|}{|I_{1,1}|} \left(1 - \frac{1}{|I_{1,1} \cap \mathcal{X}_n|} - \frac{Ab}{h_0} \frac{|I_1|}{|I_{1,1}|} \right) \\ &\geqslant \frac{|I_{1,1} \cap \mathcal{X}_n|}{|I_{1,1}|} \left(1 - \frac{1}{|I_{1,1} \cap \mathcal{X}_n|} - \frac{c_2 \varepsilon}{24} \right). \end{aligned} \tag{2.63}$$

Next, combining (2.57) and (2.62), and recalling that $|I_1| = C/n$, we obtain

$$|I_{1,1} \cap \mathcal{X}_n| \geqslant \frac{c_2 C \varepsilon}{5Ab^2}.$$

We want the bound

$$\frac{1}{|I_{1,1}\cap\mathcal{X}_n|}\leqslant\frac{c_2\varepsilon}{24}, \tag{2.64}$$

and this can be guaranteed if we ensure that

$$C\geqslant\frac{120Ab^2}{(c_2\varepsilon)^2}.$$

Combining (2.63) and (2.64), we now obtain

$$\frac{|I_1(\star)\cap\mathcal{X}_n|}{|I_1(\star)|}\geqslant\frac{|I_{1,1}\cap\mathcal{X}_n|}{|I_{1,1}|}\left(1-\frac{c_2\varepsilon}{12}\right).$$

Combining this with (2.58), we deduce that

$$\frac{|I_1(\star)\cap\mathcal{X}_n|}{|I_1(\star)|}\geqslant\left(1+\frac{c_2\varepsilon}{4}\right)\left(1-\frac{c_2\varepsilon}{12}\right)\frac{|I_1\cap\mathcal{X}_n|}{|I_1|}\geqslant\left(1+\frac{c_2\varepsilon}{12}\right)\frac{|I_1\cap\mathcal{X}_n|}{|I_1|}, \tag{2.65}$$

provided that ϵ is sufficiently small. Thus the switch from I_1 to $I_1(\star)$ leads to density increase by a factor $1+c_2\varepsilon/12$, with the added benefit that the ratio $|I_1|/|I_1(\star)|$ is an integer h_0.

To obtain a subinterval of I_0 of the same length as $I_1(\star)$, we next divide I_0 into h_0 equal parts, and denote by $I_0(\star)$ one of these subintervals with the minimum intersection with the set $\mathcal{X}_n$. Then

$$\frac{|I_0(\star)\cap\mathcal{X}_n|}{|I_0(\star)|}\leqslant\frac{|I_0\cap\mathcal{X}_n|}{|I_0|}. \tag{2.66}$$

Note that

$$|I_0(\star)|=|I_1(\star)|=\frac{C_1}{n},\quad C_1=\frac{C}{h_0},\quad h_0<c_3(\varepsilon), \tag{2.67}$$

where the constant $c_3(\varepsilon)>0$ is independent of n and C.

3. Iteration process: step 1

Let $\mathcal{I}_n(\mathcal{P};C_1)$ denote the collection of any subinterval I of any vertical edge of $\mathcal{P}$ with length $|I|=C_1/n$, and let $I_0^{(1)},I_1^{(1)}\in\mathcal{I}_n(\mathcal{P};C_1)$ be subintervals satisfying

$$V_n(I_0^{(1)})=\min_{I\in\mathcal{I}_n(\mathcal{P};C_1)}|I\cap\mathcal{X}_n|\quad\text{and}\quad V_n(I_1^{(1)})=\max_{I\in\mathcal{I}_n(\mathcal{P};C_1)}|I\cap\mathcal{X}_n|,$$

so that $I_0^{(1)}$ and $I_1^{(1)}$ have respectively the smallest and largest visiting numbers with respect to $\mathcal{X}_n$ among all the subintervals I under consideration. It is clear that

$$|I_0^{(1)}\cap\mathcal{X}_n|\leqslant\frac{C_1}{b}\leqslant|I_1^{(1)}\cap\mathcal{X}_n|.$$

Furthermore, it either, in Case 1, follows from (2.48)–(2.50) that

$$\frac{|I_0^{(1)}\cap\mathcal{X}_n|}{|I_0^{(1)}|}\leqslant\frac{|I_0(\star)\cap\mathcal{X}_n|}{|I_0(\star)|}\leqslant\left(1-\frac{c_2\varepsilon}{16}\right)\frac{|I_0\cap\mathcal{X}_n|}{|I_0|} \tag{3.1}$$

and

$$\frac{|I_1^{(1)}\cap\mathcal{X}_n|}{|I_1^{(1)}|}\geqslant\frac{|I_1(\star)\cap\mathcal{X}_n|}{|I_1(\star)|}\geqslant\frac{|I_1\cap\mathcal{X}_n|}{|I_1|}, \tag{3.2}$$

or, in Case 2, follows from (2.65)–(2.67) that

$$\frac{|I_0^{(1)}\cap\mathcal{X}_n|}{|I_0^{(1)}|}\leqslant\frac{|I_0(\star)\cap\mathcal{X}_n|}{|I_0(\star)|}\leqslant\frac{|I_0\cap\mathcal{X}_n|}{|I_0|} \tag{3.3}$$

and

$$\frac{|I_1^{(1)}\cap\mathcal{X}_n|}{|I_1^{(1)}|}\geqslant\frac{|I_1(\star)\cap\mathcal{X}_n|}{|I_1(\star)|}\geqslant\left(1+\frac{c_2\varepsilon}{12}\right)\frac{|I_1\cap\mathcal{X}_n|}{|I_1|}. \tag{3.4}$$

Combining (2.3) with (3.1) and (3.2), or with (3.3) and (3.4), we obtain

$$\frac{|I_0^{(1)}\cap\mathcal{X}_n|}{|I_1^{(1)}\cap\mathcal{X}_n|}<1-\varepsilon. \tag{3.5}$$

Remark. Note that (3.5) is the analog of (2.3) and Case B in Step 0. It follows that if Case B in Step 0 holds, then there is no analog of Case A in Step 1.

We divide the interval $I_0^{(1)}$ into 3 subintervals of equal length, and denote by $I_0^{(1)}(0)$ one of them with the minimum intersection with the set $\mathcal{X}_n$, so that

$$\frac{|I_0^{(1)}(0)\cap\mathcal{X}_n|}{|I_0^{(1)}(0)|}\leqslant\frac{|I_0^{(1)}\cap\mathcal{X}_n|}{|I_0^{(1)}|}.$$

The α-flow moves $I_0^{(1)}(0)$ forward. Repeating the argument in Step 0 between (2.4) and (2.22) with I_0,I_1,C replaced by $I_0^{(1)},I_1^{(1)},C_1$, we obtain a subinterval $I_1^{(1*)}(j_0^{(1)})\subset I_1^{(1)}$ such that

$$\frac{1}{3}|I_1^{(1)}|\geqslant|I_1^{(1*)}(j_0^{(1)})|\geqslant\frac{c_2C_1}{n}=c_2|I_1^{(1)}|, \tag{3.6}$$

the analog of (2.21). Note that $j_0^{(1)}$ may be different from j_0, but it still satisfies the upper bound c_1+1, where the constant c_1 is given by (2.20). It then follows from (2.22) that the constant c_2 in (3.6) is exactly the same as before.

We have two cases:

Case 1. We have

$$\frac{|I_1^{(1*)}(j_0^{(1)})\cap\mathcal{X}_n|}{|I_1^{(1*)}(j_0^{(1)})|}\geqslant\left(1-\frac{\varepsilon}{2}\right)\frac{|I_1^{(1)}\cap\mathcal{X}_n|}{|I_1^{(1)}|}. \tag{3.7}$$

Case 2. We have

$$\frac{|I_1^{(1*)}(j_0^{(1)})\cap\mathcal{X}_n|}{|I_1^{(1*)}(j_0^{(1)})|}<\left(1-\frac{\varepsilon}{2}\right)\frac{|I_1^{(1)}\cap\mathcal{X}_n|}{|I_1^{(1)}|}. \tag{3.8}$$

3.1. **Case 1: density decrease.** Suppose that the inequality (3.7) holds. Then an argument analogous to that in Step 0 between (2.23) and (2.37) now leads to the inequality

$$|I_0^{(1*)}(j_0^{(1)})\cap\mathcal{X}_n|\geqslant\left(1-\frac{3\varepsilon}{4}\right)|I_0^{(1*)}(j_0^{(1)})|\frac{|I_1^{(1)}\cap\mathcal{X}_n|}{|I_1^{(1)}|}, \tag{3.9}$$

provided that

$$C_1\geqslant\frac{1728A(A+1)^3c_0b}{\varepsilon}\quad\text{and}\quad C_1\geqslant\frac{8b}{c_2\varepsilon}.$$

Corresponding to (2.38), there are at most two subintervals $I_{0,1}^{(1)},I_{0,2}^{(1)}\subset I_0^{(1)}$ such that

$$I_0^{(1)}=I_0^{(1*)}(j_0^{(1)})\cup I_{0,1}^{(1)}\cup I_{0,2}^{(1)}.$$

An argument analogous to that in Step 0 between (2.38) and (2.43) then shows that, without loss of generality,

$$|I_{0,1}^{(1)}|\geqslant\frac{c_2\varepsilon}{8}|I_0^{(1)}|,$$

as well as

$$\frac{|I_{0,1}^{(1)} \cap \mathcal{X}_n|}{|I_{0,1}^{(1)}|} \leqslant \left(1 - \frac{c_2\varepsilon}{8}\right) \frac{|I_0^{(1)} \cap \mathcal{X}_n|}{|I_0^{(1)}|}.$$

An argument analogous to that in Step 0 between (2.45) and (2.48) then leads to the existence of a subinterval $I_0^{(1)}(\star) \subset I_0^{(1)}$ satisfying $|I_0^{(1)}|/|I_0^{(1)}(\star)| = h_1$, where h_1 is the unique integer satisfying

$$h_1 - 1 < \frac{16|I_0^{(1)}|}{c_2\varepsilon|I_{0,1}^{(1)}|} \leqslant h_1,$$

such that

$$\frac{|I_0^{(1)}(\star) \cap \mathcal{X}_n|}{|I_0^{(1)}(\star)|} \leqslant \left(1 - \frac{c_2\varepsilon}{16}\right) \frac{|I_0^{(1)} \cap \mathcal{X}_n|}{|I_0^{(1)}|}. \tag{3.10}$$

To obtain a subinterval of $I_1^{(1)}$ of the same length as $I_0^{(1)}(\star)$, we next divide $I_1^{(1)}$ into h_1 equal parts, and denote by $I_1^{(1)}(\star)$ one of these subintervals with the maximum intersection with the set $\mathcal{X}_n$. Then

$$\frac{|I_1^{(1)}(\star) \cap \mathcal{X}_n|}{|I_1^{(1)}(\star)|} \geqslant \frac{|I_1^{(1)} \cap \mathcal{X}_n|}{|I_1^{(1)}|}.$$

Note that

$$|I_1^{(1)}(\star)| = |I_0^{(1)}(\star)| = \frac{C_2}{n}, \quad C_2 = \frac{C_1}{h_1}, \quad h_1 < c_3(\varepsilon),$$

where the constant $c_3(\varepsilon) > 0$ is as in Step 0.

3.2. **Case 2: density increase.** Suppose that the inequality (3.8) holds. Then corresponding to (2.51), there are at most two subintervals $I_{1,1}^{(1)}, I_{1,2}^{(1)} \subset I_1^{(1)}$ such that

$$I_1^{(1)} = I_1^{(1*)}(j_0^{(1)}) \cup I_{1,1}^{(1)} \cup I_{1,2}^{(1)}. \tag{3.11}$$

An argument analogous to that in Step 0 between (2.51) and (2.58) then shows that, without loss of generality,

$$|I_{1,1}^{(1)}| \geqslant \frac{c_2\varepsilon}{5Ab}|I_1^{(1)}|,$$

as well as

$$\frac{|I_{1,1}^{(1)} \cap \mathcal{X}_n|}{|I_{1,1}^{(1)}|} \geqslant \left(1 + \frac{c_2\varepsilon}{4}\right) \frac{|I_1^{(1)} \cap \mathcal{X}_n|}{|I_1^{(1)}|}.$$

An argument analogous to that in Step 0 between (2.59) and (2.65) then leads to the existence of a subinterval $I_1^{(1)}(\star) \subset I_1^{(1)}$ satisfying $|I_1^{(1)}|/|I_1^{(1)}(\star)| = h_1$, where h_1 is the unique integer satisfying

$$h_1 - 1 < \frac{24Ab|I_1^{(1)}|}{c_2\varepsilon|I_{1,1}^{(1)}|} \leqslant h_1,$$

such that

$$\frac{|I_1^{(1)}(\star) \cap \mathcal{X}_n|}{|I_1^{(1)}(\star)|} \geqslant \left(1 + \frac{c_2\varepsilon}{12}\right) \frac{|I_1^{(1)} \cap \mathcal{X}_n|}{|I_1^{(1)}|},$$

provided that

$$C_1 \geqslant \frac{120Ab^2}{(c_2\varepsilon)^2}. \tag{3.12}$$

To obtain a subinterval of $I_0^{(1)}$ of the same length as $I_1^{(1)}(\star)$, we next divide $I_0^{(1)}$ into h_1 equal parts, and denote by $I_0^{(1)}(\star)$ one of these subintervals with the minimum intersection with the set $\mathcal{X}_n$. Then

$$\frac{|I_0^{(1)}(\star)\cap\mathcal{X}_n|}{|I_0^{(1)}(\star)|}\leqslant\frac{|I_0^{(1)}\cap\mathcal{X}_n|}{|I_0^{(1)}|}.$$

Note that

$$|I_0^{(1)}(\star)|=|I_1^{(1)}(\star)|=\frac{C_2}{n},\quad C_2=\frac{C_1}{h_1},\quad h_1<c_3(\varepsilon),$$

where the constant $c_3(\varepsilon)>0$ is as in Step 0.

4. Iteration process: general step

Let $\mathcal{I}_n(\mathcal{P};C_i)$ denote the collection of any subinterval I of any vertical edge of $\mathcal{P}$ with length $|I|=C_i/n$, and let $I_0^{(i)},I_1^{(i)}\in\mathcal{I}_n(\mathcal{P};C_i)$ be subintervals satisfying

$$V_n(I_0^{(i)})=\min_{I\in\mathcal{I}_n(\mathcal{P};C_i)}|I\cap\mathcal{X}_n|\quad\text{and}\quad V_n(I_1^{(i)})=\max_{I\in\mathcal{I}_n(\mathcal{P};C_i)}|I\cap\mathcal{X}_n|,$$

so that $I_0^{(i)}$ and $I_1^{(i)}$ have respectively the smallest and largest visiting numbers with respect to $\mathcal{X}_n$ among all the subintervals I under consideration. It is clear that

$$|I_0^{(i)}\cap\mathcal{X}_n|\leqslant\frac{C_i}{b}\leqslant|I_1^{(i)}\cap\mathcal{X}_n|.$$

Furthermore, we either, in Case 1 in the previous step and analogous to (3.1) and (3.2), have

$$\frac{|I_0^{(i)}\cap\mathcal{X}_n|}{|I_0^{(i)}|}\leqslant\left(1-\frac{c_2\varepsilon}{16}\right)\frac{|I_0^{(i-1)}\cap\mathcal{X}_n|}{|I_0^{(i-1)}|}\tag{4.1}$$

and

$$\frac{|I_1^{(i)}\cap\mathcal{X}_n|}{|I_1^{(i)}|}\geqslant\frac{|I_1^{(i-1)}\cap\mathcal{X}_n|}{|I_1^{(i-1)}|},\tag{4.2}$$

or, in Case 2 in the previous step and analogous to (3.3) and (3.4), have

$$\frac{|I_0^{(i)}\cap\mathcal{X}_n|}{|I_0^{(i)}|}\leqslant\frac{|I_0^{(i-1)}\cap\mathcal{X}_n|}{|I_0^{(i-1)}|}\tag{4.3}$$

and

$$\frac{|I_1^{(i)}\cap\mathcal{X}_n|}{|I_1^{(i)}|}\geqslant\left(1+\frac{c_2\varepsilon}{12}\right)\frac{|I_1^{(i-1)}\cap\mathcal{X}_n|}{|I_1^{(i-1)}|}.\tag{4.4}$$

Combining the estimate

$$\frac{|I_0^{(i-1)}\cap\mathcal{X}_n|}{|I_1^{(i-1)}\cap\mathcal{X}_n|}<1-\varepsilon$$

from the previous step with (4.1) and (4.2), or with (4.3) and (4.4), we obtain

$$\frac{|I_0^{(i)}\cap\mathcal{X}_n|}{|I_1^{(i)}\cap\mathcal{X}_n|}<1-\varepsilon,$$

the analog of (2.3) and (3.5).

On the other hand, iterating (4.1)–(4.4) carefully, we obtain

$$\frac{|I_0^{(i)}\cap\mathcal{X}_n|}{|I_0^{(i)}|}\leqslant\left(1-\frac{c_2\varepsilon}{16}\right)^{i_1}\frac{|I_0\cap\mathcal{X}_n|}{|I_0|}\tag{4.5}$$

and

$$\frac{|I_1^{(i)} \cap \mathcal{X}_n|}{|I_1^{(i)}|} \geqslant \left(1 + \frac{c_2\varepsilon}{12}\right)^{i_2} \frac{|I_1 \cap \mathcal{X}_n|}{|I_1|}, \tag{4.6}$$

where i_1 and i_2 denote respectively the number of times Case 1 and Case 2 are valid in the previous $i = i_1 + i_2$ steps. Combining (2.1), (4.5) and (4.6), and recalling that $|I_0| = |I_1|$ and $|I_0^{(i)}| = |I_1^{(i)}|$, we obtain the inequality

$$|I_0^{(i)} \cap \mathcal{X}_n| \leqslant \left(1 - \frac{c_2\varepsilon}{16}\right)^{i_1} |I_1^{(i)} \cap \mathcal{X}_n|. \tag{4.7}$$

We divide the interval $I_0^{(i)}$ into 3 subintervals of equal length, and denote by $I_0^{(i)}(0)$ one of them with the minimum intersection with the set $\mathcal{X}_n$. The α-flow moves $I_0^{(i)}(0)$ forward. Corresponding to (2.21) and (3.6), we obtain a subinterval $I_1^{(i*)}(j_0^{(i)}) \subset I_1^{(i)}$ such that

$$\frac{1}{3}|I_1^{(i)}| \geqslant |I_1^{(i*)}(j_0^{(i)})| \geqslant \frac{c_2 C_i}{n} = c_2|I_1^{(i)}|, \tag{4.8}$$

where the constant c_2 in (4.8) is exactly the same as before.

We have two cases:

Case 1. We have

$$\frac{|I_1^{(i*)}(j_0^{(i)}) \cap \mathcal{X}_n|}{|I_1^{(i*)}(j_0^{(i)})|} \geqslant \left(1 - \frac{\varepsilon}{2}\right) \frac{|I_1^{(i)} \cap \mathcal{X}_n|}{|I_1^{(i)}|}. \tag{4.9}$$

Case 2. We have

$$\frac{|I_1^{(i*)}(j_0^{(i)}) \cap \mathcal{X}_n|}{|I_1^{(i*)}(j_0^{(i)})|} < \left(1 - \frac{\varepsilon}{2}\right) \frac{|I_1^{(i)} \cap \mathcal{X}_n|}{|I_1^{(i)}|}. \tag{4.10}$$

4.1. **Case 1: density decrease.** Suppose that the inequality (4.9) holds. Then an argument analogous to that in Step 0 between (2.23) and (2.48) and that in Step 1 between (3.9) and (3.10) leads to the existence of a subinterval $I_0^{(i)}(\star) \subset I_0^{(i)}$ satisfying $|I_0^{(i)}|/|I_0^{(i)}(\star)| = h_i$, where h_i is an integer and

$$\frac{|I_0^{(i)}(\star) \cap \mathcal{X}_n|}{|I_0^{(i)}(\star)|} \leqslant \left(1 - \frac{c_2\varepsilon}{16}\right) \frac{|I_0^{(i)} \cap \mathcal{X}_n|}{|I_0^{(i)}|}.$$

provided that

$$C_i \geqslant \frac{1728A(A+1)^3 c_0 b}{\varepsilon} \quad \text{and} \quad C_i \geqslant \frac{8b}{c_2\varepsilon}. \tag{4.11}$$

To obtain a subinterval of $I_1^{(i)}$ of the same length as $I_0^{(i)}(\star)$, we next divide $I_1^{(i)}$ into h_i equal parts, and denote by $I_1^{(i)}(\star)$ one of these subintervals with the maximum intersection with the set $\mathcal{X}_n$. Then

$$\frac{|I_1^{(i)}(\star) \cap \mathcal{X}_n|}{|I_1^{(i)}(\star)|} \geqslant \frac{|I_1^{(i)} \cap \mathcal{X}_n|}{|I_1^{(i)}|}.$$

We have

$$|I_1^{(i)}(\star)| = |I_0^{(i)}(\star)| = \frac{C_{i+1}}{n}, \quad C_{i+1} = \frac{C_i}{h_i}, \quad h_i < c_3(\varepsilon), \tag{4.12}$$

where the constant $c_3(\varepsilon) > 0$ is as in Step 0.

4.2. **Case 2: density increase.** Suppose that the inequality (4.10) holds. Then an argument analogous to that in Step 0 between (2.51) and (2.65) and that in Step 1 between (3.11) and (3.12) leads to the existence of a subinterval $I_1^{(i)}(\star) \subset I_1^{(i)}$ satisfying $|I_1^{(i)}|/|I_1^{(i)}(\star)| = h_i$, where h_i is an integer and

$$\frac{|I_1^{(i)}(\star) \cap \mathcal{X}_n|}{|I_1^{(i)}(\star)|} \geqslant \left(1 + \frac{c_2\varepsilon}{12}\right) \frac{|I_1^{(i)} \cap \mathcal{X}_n|}{|I_1^{(i)}|},$$

provided that

$$C_i \geqslant \frac{120Ab^2}{(c_2\varepsilon)^2}. \tag{4.13}$$

To obtain a subinterval of $I_0^{(i)}$ of the same length as $I_1^{(i)}(\star)$, we next divide $I_0^{(i)}$ into h_i equal parts, and denote by $I_0^{(i)}(\star)$ one of these subintervals with the minimum intersection with the set $\mathcal{X}_n$. Then

$$\frac{|I_0^{(i)}(\star) \cap \mathcal{X}_n|}{|I_0^{(i)}(\star)|} \leqslant \frac{|I_0^{(i)} \cap \mathcal{X}_n|}{|I_0^{(i)}|}.$$

We have

$$|I_0^{(i)}(\star)| = |I_1^{(i)}(\star)| = \frac{C_{i+1}}{n}, \quad C_{i+1} = \frac{C_i}{h_i}, \quad h_i < c_3(\varepsilon), \tag{4.14}$$

where the constant $c_3(\varepsilon) > 0$ is as in Step 0.

5. Iteration process: deriving a contradiction

We now attempt to derive the necessary contradiction.

Suppose first that Case 1 holds in Step i.

Corresponding to (2.35) and (3.9), we have the inequality

$$|I_0^{(i*)}(j_0^{(i)}) \cap \mathcal{X}_n| \geqslant \left(1 - \frac{3\varepsilon}{4}\right) |I_0^{(i*)}(j_0^{(i)})| \frac{|I_1^{(i)} \cap \mathcal{X}_n|}{|I_1^{(i)}|}, \tag{5.1}$$

provided that

$$C_i \geqslant \frac{1728A(A+1)^3c_0b}{\varepsilon} \quad \text{and} \quad C_i \geqslant \frac{8b}{c_2\varepsilon}.$$

Clearly $I_0^{(i*)}(j_0^{(i)}) \subset I_0^{(i)}$, so it follows from (4.7) that

$$|I_0^{(i*)}(j_0^{(i)}) \cap \mathcal{X}_n| \leqslant \left(1 - \frac{c_2\varepsilon}{16}\right)^{i_1} |I_1^{(i)} \cap \mathcal{X}_n|. \tag{5.2}$$

On the other hand, combining (4.8) with (5.1) leads to the inequality

$$|I_0^{(i*)}(j_0^{(i)}) \cap \mathcal{X}_n| \geqslant c_2 \left(1 - \frac{3\varepsilon}{4}\right) |I_1^{(i)} \cap \mathcal{X}_n|. \tag{5.3}$$

Clearly (5.2) and (5.3) contradict each other if

$$\left(1 - \frac{c_2\varepsilon}{16}\right)^{i_1} < \frac{c_2}{2},$$

noting that $0 < \varepsilon < 1/2$. This gives an upper bound $c_4(\varepsilon)$ to i_1, the number of times that Case 1 holds among the first i steps.

Suppose next that Case 2 holds in Step i.

Combining (2.1) and (4.6), and noting that $|I_1| = C/n$, we have

$$|I_1^{(i)} \cap \mathcal{X}_n| \geqslant \left(1 + \frac{c_2\varepsilon}{12}\right)^{i_2} \frac{n}{b} |I_1^{(i)}|. \tag{5.4}$$

On the other hand, it follows from (2.31) that

$$|I_1^{(i)} \cap \mathcal{X}_n| \leqslant An|I_1^{(i)}| + 1. \tag{5.5}$$

Clearly (5.4) and (5.5) contradict each other if

$$\left(1+\frac{c_2\varepsilon}{12}\right)^{i_2} > 2Ab,$$

provided that $C_i \geqslant 1$. This gives an upper bound $c_5(\varepsilon)$ to i_2, the number of times that Case 2 holds among the first i steps.

If $i > c_4(\varepsilon) + c_5(\varepsilon)$, then neither Case 1 nor Case 2 in Step i can be valid. To show that Case B is impossible, it remains to analyze the various constants in our argument.

Recall that the constants c_0 and c_2 depend only on $\mathcal{P}$ and α, and are independent of n, C and ε, while the constant A depends only on α, and the constant b depends only on $\mathcal{P}$. It remains to study the constants C_i, which must satisfy

$$C_i = \frac{C}{h_0 h_1, \ldots, h_{i-1}} \geqslant \max\left\{\frac{1728A(A+1)^3 c_0 b}{\varepsilon}, \frac{8b}{c_2\varepsilon}, \frac{120Ab^2}{(c_2\varepsilon)^2}\right\}, \tag{5.6}$$

in view of (4.11)–(4.14) and $C_i \geqslant 1$. Since $h_i < c_3(\varepsilon)$ and the iteration process must stop after at most $c_6(\varepsilon) = c_4(\varepsilon) + c_5(\varepsilon)$ steps, it follows that (5.6) is satisfied provided that C is chosen sufficiently large in terms of $\mathcal{P}, \alpha$ and ε.

6. Proof of Theorem 1

We have already shown that Case B leads to a contradiction. To complete the proof of Theorem 1, it remains to investigate Case A, when the inequality (2.2) holds. We have the following almost trivial observation.

Lemma 3. *Suppose that J is a subinterval of any vertical edge of $\mathcal{P}$ with length $|J| \geqslant 3C/n$. If (2.2) holds, then*

$$(1-\varepsilon)\left(\frac{|J|}{|I_1|}-3\right)|I_1 \cap \mathcal{X}_n| \leqslant |J \cap \mathcal{X}_n| \leqslant \left(\frac{|J|}{|I_1|}+3\right)|I_1 \cap \mathcal{X}_n|. \tag{6.1}$$

Proof. Let $k = [n/C]$ denote the integer part of n/C. Then we can split any vertical edge of $\mathcal{P}$ into a union of k special subintervals of length C/n and an extra short interval with length w satisfying $0 \leqslant w < C/n$ at the top end of the vertical edge.

Consider the unique integer ℓ_0 that satisfies the inequalities

$$\ell_0 \leqslant \frac{|J|}{C/n} = \frac{|J|}{|I_1|} < \ell_0 + 1. \tag{6.2}$$

Then J contains at least $\ell_0 - 2$ of these special subintervals of length C/n. Combining this observation with (2.2) and the second inequality in (6.2) leads to the lower bound

$$|J \cap \mathcal{X}_n| \geqslant (\ell_0 - 2)|I_0 \cap \mathcal{X}_n| > \left(\frac{|J|}{|I_1|}-3\right)(1-\varepsilon)|I_1 \cap \mathcal{X}_n|.$$

On the other hand, J is covered by $\ell_0 + 2$ special subintervals of length C/n and the extra short subinterval of length w which is contained in a subinterval of the vertical edge of length C/n. Combining this observation with the first inequality in (6.2) leads to the upper bound

$$|J \cap \mathcal{X}_n| \leqslant (\ell_0 + 3)|I_1 \cap \mathcal{X}_n| \leqslant \left(\frac{|J|}{|I_1|}+3\right)|I_1 \cap \mathcal{X}_n|.$$

This completes the proof. □

Let $|J| = 3|I_1|/\varepsilon = 3C/\varepsilon n$. Then

$$\left| V_n(J) - \frac{n|J|}{b} \right| = \left| |J \cap \mathcal{X}_n| - \frac{n}{b}\frac{3|I_1|}{\varepsilon} \right|$$
$$\leqslant \left| |J \cap \mathcal{X}_n| - \frac{3}{\varepsilon}|I_1 \cap \mathcal{X}_n| \right| + \frac{3|I_1|}{\varepsilon}\left(\frac{|I_1 \cap \mathcal{X}_n|}{|I_1|} - \frac{n}{b} \right). \tag{6.3}$$

With $|J| = 3|I_1|/\varepsilon$ in (6.1), we have

$$\frac{3(1-\varepsilon)^2}{\varepsilon}|I_1 \cap \mathcal{X}_n| \leqslant |J \cap \mathcal{X}_n| \leqslant \frac{3(1+\varepsilon)}{\varepsilon}|I_1 \cap \mathcal{X}_n|,$$

and this implies

$$\left| |J \cap \mathcal{X}_n| - \frac{3}{\varepsilon}|I_1 \cap \mathcal{X}_n| \right| \leqslant 6|I_1 \cap \mathcal{X}_n|. \tag{6.4}$$

On the other hand, it is clear from (2.1) and (2.2) that

$$\frac{|I_1 \cap \mathcal{X}_n|}{|I_1|} \geqslant \frac{n}{b} \geqslant \frac{|I_0 \cap \mathcal{X}_n|}{|I_0|} \geqslant (1-\varepsilon)\frac{|I_1 \cap \mathcal{X}_n|}{|I_1|}. \tag{6.5}$$

It then follows from (6.5) that

$$\frac{|I_1 \cap \mathcal{X}_n|}{|I_1|} - \frac{n}{b} \leqslant \frac{|I_1 \cap \mathcal{X}_n|}{|I_1|} - \frac{|I_0 \cap \mathcal{X}_n|}{|I_0|} \leqslant \frac{|I_1 \cap \mathcal{X}_n|}{|I_1|} - (1-\varepsilon)\frac{|I_1 \cap \mathcal{X}_n|}{|I_1|},$$

so that

$$\frac{3|I_1|}{\varepsilon}\left(\frac{|I_1 \cap \mathcal{X}_n|}{|I_1|} - \frac{n}{b} \right) \leqslant 3|I_1 \cap \mathcal{X}_n|. \tag{6.6}$$

It also follows from (6.5) that

$$|I_1 \cap \mathcal{X}_n| \leqslant \frac{|I_1|}{1-\varepsilon}\frac{n}{b}. \tag{6.7}$$

Substituting (6.4), (6.6) and (6.7) into (6.3), we conclude that

$$\left| V_n(J) - \frac{n|J|}{b} \right| \leqslant \frac{3\varepsilon}{1-\varepsilon}\frac{n|J|}{b}. \tag{6.8}$$

Since n and J are arbitrary, the inequality (6.8) proves super-micro-uniformity with $3\varepsilon(1-\varepsilon)^{-1}$ instead of ε.

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